Recent Progress on Hill’s Conjecture

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Preliminaries – Drawings

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![Diagram showing a path passing through vertices](image)
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- A drawing is called **$x$-monotone** if edges are $x$-monotone curves.
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All drawings with minimum number of crossings are simple.
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**Observation**

All drawings with minimum number of crossings are simple.

The **monotone crossing number** $\text{mon-cr}(G)$ of $G$ is the minimum number of crossings $\text{cr}(D)$ in $D$ taken over all x-monotone drawings $D$ of $G$. 
Crossing Number of $K_n$
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Conjecture (Hill, 1958)

We have $\text{cr}(K_n) = Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$ for every $n \in \mathbb{N}$. 
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**Hill’s optimal drawing of $K_{10}$:**

**Optimal 2-page drawing of $K_{10}$:**

- A drawing is **2-page** if the vertices are placed on a line $\ell$ and each edge is fully contained in a halfspace determined by $\ell$. 
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**Theorem (B., Fulek, Kynčl, 2013)**

For every \( n \in \mathbb{N} \) we have \( \text{mon-cr}(K_n) = Z(n) \).
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  - weakly semisimple \( s \)-shellable drawings.
- Since 2-page drawings are \( x \)-monotone, we have \( \text{mon-cr}(K_n) \leq Z(n) \).
Sketch of the Proof: Double Counting
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Lemma

For a simple drawing $D$ of $K_n$ we get $cr(D) = 3\left(\frac{n}{4}\right) - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n - 2 - k)E_k(D)$. 
Sketch of the Proof: Main Trick
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- **Trick**: we estimate the sums $E_{\leq k}(D)$ defined as

$$E_{\leq k}(D) := \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k} (k + 1 - i)E_i(D).$$
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**Lemma**

For every simple drawing $D$ of $K_n$ we have

$$cr(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2} \left( 1 + (-1)^n \right) E_{\leq \lfloor n/2 \rfloor - 2}(D).$$
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- That is, we want a lower bound for \( E_{\leq k}(D) \).
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- For a simple $x$-monotone drawing $D$ of $K_n$ let $D'$ be $D$ with the rightmost vertex removed.
- A $k$-edge in $D$ is a $(D, D')$-invariant $k$-edge if it is a $k$-edge in $D'$. 
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Let \( E_k(D, D') \) be the number of \((D, D')\)-invariant \( k \)-edges.

Let \( E_{\leq k}(D, D') \) be the sum \( \sum_{i=0}^{k} E_k(D, D') \).
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**Lemma**

For a simple $x$-monotone $D$ we have $E_{\le k}(D, D') \ge \sum_{i=1}^{k+1} (k + 2 - i) = \binom{k+2}{2}$. 
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- For $0 \leq k \leq (n - 3)/2$ and every $i \in [k + 1]$, the $k + 2 - i$ bottommost and $k + 2 - i$ topmost right edges at $v_i$ are $j$-edges for some $j \leq k$. 
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Sketch of the Proof: Final Bound

**Theorem**

Let $n \geq 3$ and let $D$ be a simple $x$-monotone drawing of $K_n$. Then for every $k$, $0 \leq k < n/2 - 1$, we have $E_{\leq k}(D) \geq 3^{(k+3)/3}$. 
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- Proceed by induction on $n$ and $k$.
- Edges incident to the rightmost vertex contribute to $E_{\leq k}(D)$ by

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- An $i$-edge, $i \leq k$, in $D'$ contributes by $k - i$ to $E_{\leq k-1}(D')$ and by $k - i$ or $k - i + 1$ to $E_{\leq k}(D)$.
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  E_{\leq k}(D) = 2 \binom{k + 2}{2} + E_{\leq k-1}(D') + E_{\leq k}(D, D')
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  \geq 3 \binom{k+3}{3} - \binom{k+2}{2} + E_{\leq k}(D, D') \geq 3 \binom{k+3}{3}.
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Characterization of Pseudolinear and $x$-monotone Drawings
Use the orientations of triangles to characterize $x$-monotone drawings of $K_n$. 

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Color each triangle with a sign \( + \) or \( - \) according to its orientation \( \Rightarrow \) a signature of \( D \).
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- Use the orientations of triangles to characterize $x$-monotone drawings of $K_n$.
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All 16 possible forms of 4-tuples:

- $-+-+-$
- $+---$
- $--++$
- $---+$

Pseudolinear

- $++++$
- $---+$
- $++-+$
- $+-++$

Semisimple $x$-monotone

- $++++$
- $-+++$
- $++-+$
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- All 16 possible forms of 4-tuples:

![Diagram showing different orientations and forms of 4-tuples]

- Pseudolinear
- Semisimple $x$-monotone
- Simple $x$-monotone
General Drawings
There are (optimal) drawings of $K_n$ where $E_{\leq k}(D) \geq 3\binom{k+3}{3}$ does not hold.
There are (optimal) drawings of $K_n$ where $E_{\leq k}(D) \geq 3^{k+3} \choose 3$ does not hold.
There are (optimal) drawings of $K_n$ where $E_{\leq \leq k}(D) \geq 3^{(k+3)}$ does not hold.
General Drawings

- There are (optimal) drawings of $K_n$ where $E_{\leq k}(D) \geq 3\left(\frac{k+3}{3}\right)$ does not hold.

- Here we have $E_0 = 5$ and $E_1 = 0$, hence $E_{\leq 1} = 10 < 12 = 3\left(\frac{1+3}{3}\right)$. 

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\includegraphics[width=0.4\textwidth]{drawing1.png}
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Let $n \geq 3$ and let $D$ be a simple drawing of $K_n$. Then for every $k$ satisfying $0 \leq k < n/2 - 1$, we have $E_{\leq \leq \leq k}(D) \geq 3\binom{k+4}{4}$. 
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- Implies Hill’s conjecture. All drawings we have found satisfy this conjecture.