Algorithmic game theory – Tutorial 5*

December 16, 2021

1 Revenue-maximizing auctions

We consider the *Bayesian model*, which consists of a single-parameter environment (x, p) with n bidders, where, for each bidder i, the private valuation v_i of i is drawn from a probability distribution F_i with density function f_i and with support contained in $[0, v_{max}]$. The distributions $F_1, \dots, F_n$ are independent, but not necessarily the same. We recall that if F is a probability distribution with density f and with support $[0, v_{max}]$, then $f(z) = \frac{\mathrm{d}}{\mathrm{d}z} F(z)$ and $F(x) = \int_0^x f(z)\,\mathrm{d}z$. Also, for a random variable X, we have $\mathbb{E}_{z \sim F}[X(z)] = \int_0^{v_{max}} X(z) \cdot f(z)\,\mathrm{d}z$.

The *virtual valuation* of bidder i with valuation v_i drawn from F_i is $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$. The *virtual social surplus* is $\sum_{i=1}^n \varphi_i(v_i) \cdot x_i(v)$.

We consider only DSIC auctions.

Exercise 1. *Let F be the uniform probability distribution on $[0, 1]$. Consider a single-item auction with two bidders 1 and 2 that have probability distributions $F_1 = F$ and $F_2 = F$ on their valuations. Prove that the expected revenue obtained by the Vickrey auction (with no reserve) is $1/3$.*

Exercise 2. *Compute the virtual valuation function of the following probability distributions and show which of these distributions are regular (meaning the virtual valuation function is strictly increasing).*

(a) *The uniform distribution $F(z) = z/a$ on $[0, a]$ with $a > 0$,*

(b) *The exponential distribution $F(z) = 1 - e^{-\lambda z}$ with rate $\lambda > 0$ on $[0, \infty)$,*

Exercise 3. *Consider a single-item auction where bidder i draws his valuation from his own regular distribution F_i, that is, the probability distributions $F_1, \dots, F_n$ can be different but all virtual valuation functions $\varphi_1, \dots, \varphi_n$ are strictly increasing.*

(a) *Give a formula for the winner's payment in an optimal auction, in terms of the bidders' virtual valuation functions φ_i. Verify that if $F_1 = \cdots = F_n$ are uniform probability distributions on $[0, 1]$, then your formula yields Vickrey auction with reserve price $1/2$.*

(b) *Find an example of an optimal auction in which the highest bidder does not win, even if he has a positive virtual valuation.*

Hint: It suffices to consider two bidders with valuations from different uniform distributions.

We assume that the bidders $1, \dots, n$ are sorted in the order $<$ so that $\frac{b_1}{w_1} \geq \cdots \geq \frac{b_n}{w_n}$. Consider the following *greedy allocation rule* $x^G = (x_1^G, \dots, x_n^G) \in X$, which for given bids $b = (b_1, \dots, b_n)$ selects a subset of bidders so that $\sum_{i=1}^n x_i^G w_i \leq W$ using the following procedure.

1. Pick winners in the order $<$ until one does not fit and then halt.
2. Return either the solution from the first step or the highest bidder, whichever creates more social surplus.

Exercise 4. *Prove that the Knapsack auction allocation rule x^G induced by the greedy $(1/2)$-approximation algorithm covered in the lecture is monotone.*

*Information about the course can be found at http://kam.mff.cuni.cz/~balko/