# Automorphism Groups of Interval Graphs

Peter Zeman
joint work with Pavel Klavík

Faculty of Mathematics and Physics,
Charles University in Prague

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## The Automorphism Group of a Disconnected Graph

- If a graph $G$ has $n$ pairwise nonisomorphic connected components $G_1, \ldots, G_n$, then

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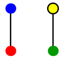
- $\mathrm{Aut}\Big($  $\Big) \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$

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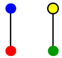
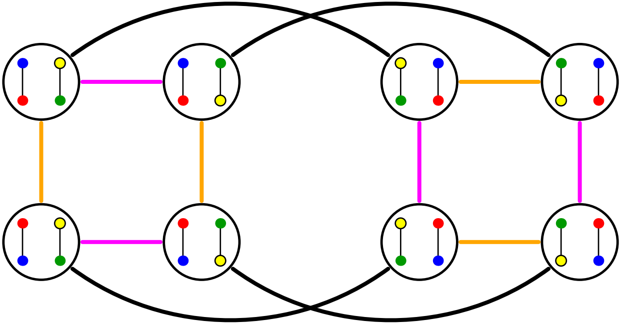
- $\mathrm{Aut}\Bigg($  $\Bigg) \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$



# Wreath product

- If a graph $G$ contains $k$ copies of $H$, then the automorphism group of $G$ is isomorphic to $\mathrm{Aut}(G) \wr \mathbb{S}_k$, where

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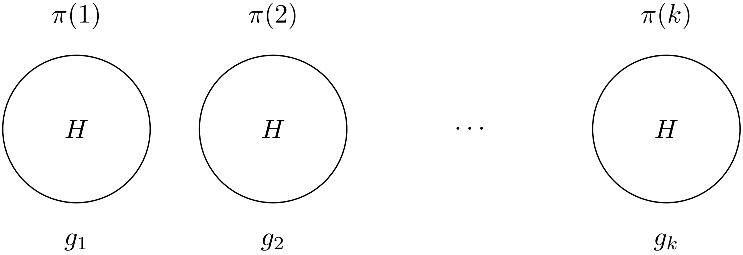

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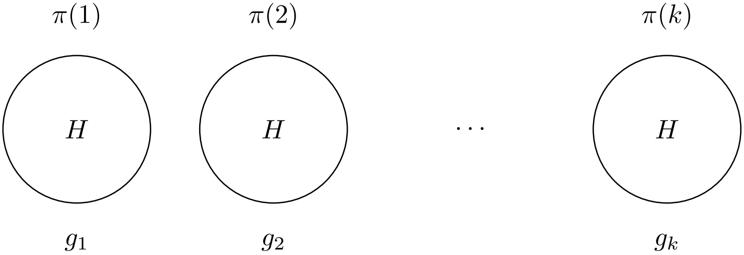
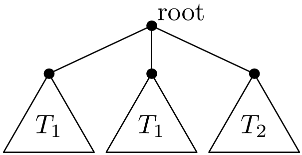



- If a graph $G$ contains $k_i$ copies of $G_i$ for $i = 1, \ldots, n$, then the automorphism group of $G$ is isomorphic to

$$\mathrm{Aut}(G_1) \wr \mathbb{S}_{k_1} \times \cdots \times \mathrm{Aut}(G_n) \wr \mathbb{S}_{k_n}.$$

# Automorphism Groups of Trees



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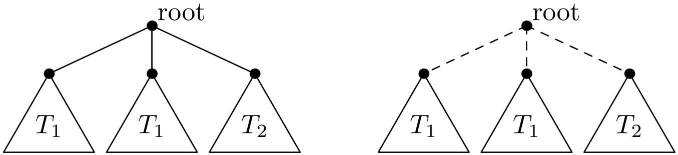


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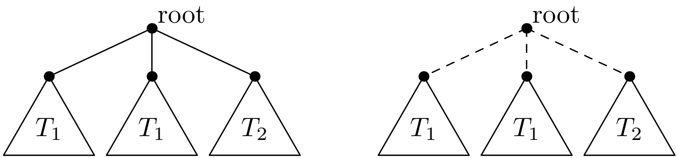


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## Theorem (Jordan, 1869)

*The finite group $\Gamma$ is isomorophic to the automorphism group of a finite tree if and only if $\Gamma \in \mathcal{T}$, where the class $\mathcal{T}$ of finite groups is defined inductively as follows:*

(a) $\{1\} \in \mathcal{T}$,

(b) *if* $\Gamma_1, \Gamma_2 \in \mathcal{T}$ *then* $\Gamma_1 \times \Gamma_2 \in \mathcal{T}$,

(c) *if* $\Gamma \in \mathcal{T}$ *and* $n \geq 2$ *then* $\Gamma \wr \mathbb{S}_n \in \mathcal{T}$.

For interval graphs we show that we need to add an operation (d).

# Interval Graphs

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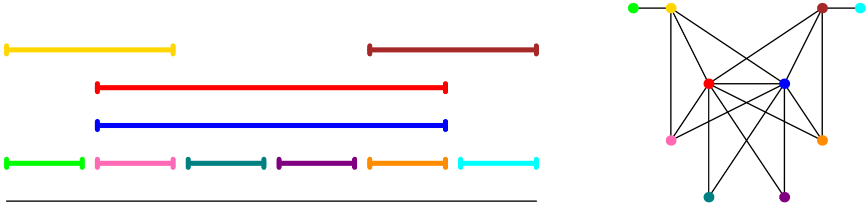

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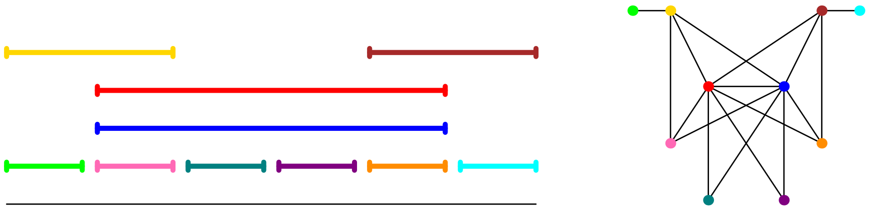

Colbourn and Kellogg found (1981) a linear time algorithm for finding a set of generators of the automorphism group of an interval graph.

# Characterization of interval graphs

**Theorem (Fulkerson and Gross)**

*A graph $G$ is an interval graph if and only if there exists and ordering of the maximal cliques such that for every vertex $v \in V(G)$, the cliques containing $v$ appear in it consequtively.*

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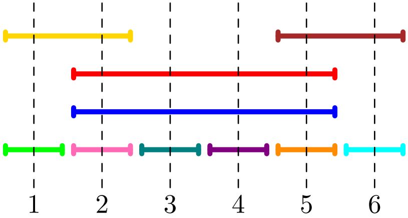


Restricting conditions for the ordering are $\{1, 2\}$, $\{5, 6\}$ and $\{2, 3, 4, 5\}$.

# PQ-trees

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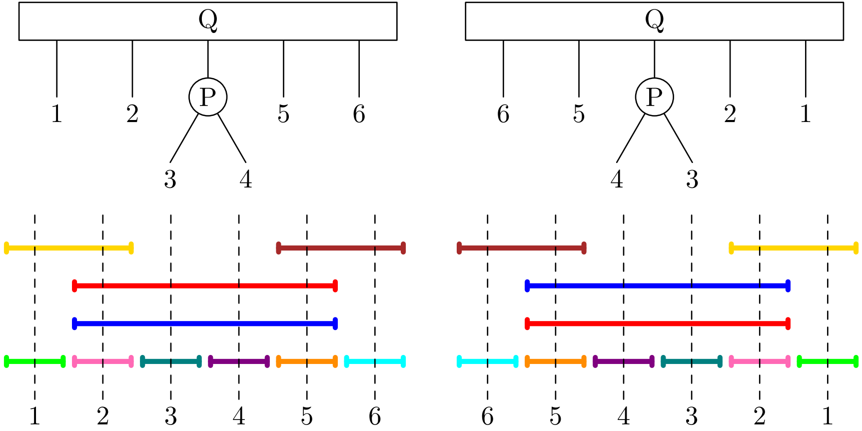

# Automorphisms of PQ-trees

Two PQ-trees $T$ and $T'$ are equivalent if one can be obtained from the other by applying the following two equivalence transformations:

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- Reverse the children of a Q-node.

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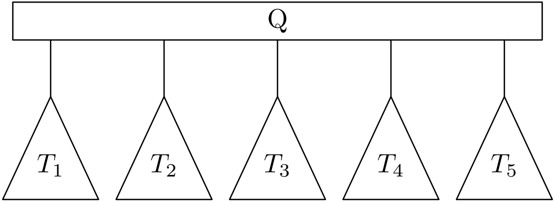
If $\varepsilon$ represents a sequence of equivalence transformations, $\varepsilon \in \mathrm{Aut}(T)$ if there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(T_\varepsilon)$ is $T$.

# Automorphism groups of PQ-trees

If we consider only PQ-trees with no Q-node, we get the same automorphism groups as for trees.




If $T_1$ is isomorphic to $T_5$ and $T_2$ is isomorphic to $T_4$, then reversing the ordering of $T_1, \ldots, T_5$ is an automorphism of $T$.

$$\begin{aligned} \mathrm{Aut}(T) &= (\mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_5)) \rtimes \mathbb{Z}_2 \\ &= \{(t_1, \ldots, t_5, z) \colon t_i \in \mathrm{Aut}(T_i), z \in \mathbb{Z}_2\} \end{aligned}$$

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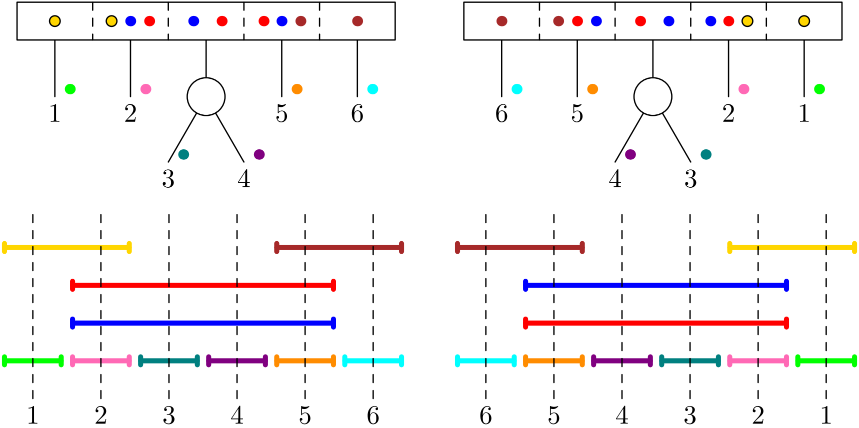

# Automorphisms of MPQ-trees

Two MPQ-trees $T$ and $T'$ are equivalent if one can be obtained from the other by applying the equivalence transformations and reordering the sections with a node preserving permutation.

If $\varepsilon$ represents a sequence of equivalence transformations and $\nu$ is a node preserving permutation, then $(\varepsilon, \nu) \in \mathrm{Aut}(T)$ if there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(T_{\varepsilon,\nu})$ is $T$.

# Automorphism groups of MPQ-trees

- If $T$ is a MPQ-tree for an interval graph $G$, then

$$\mathrm{Aut}(T) \cong \mathrm{Aut}(G).$$

- $\mathrm{Aut}(T) = E \times N$, where $E$ is the automorphism group of the corresponding PQ-tree and $N$ is a direct product of symmetric groups.

### Theorem

*The finite group $\Gamma$ is isomorophic to the automorphism group of a finite tree if and only if $\Gamma \in \mathcal{I}$, where the class $\mathcal{I}$ of finite groups is defined inductively as follows:*

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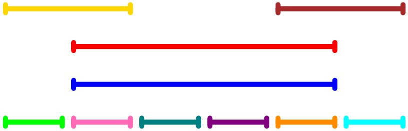
(d) *if* $\Gamma_1, \ldots, \Gamma_n \in \mathcal{I}$, $n \geq 3$ *and* $G_i$ *is the graph for which* $\mathrm{Aut}(G_i) = \Gamma_i$, *then* $(\Gamma_1 \times \cdots \times \Gamma_n) \rtimes \mathbb{Z}_2 \in \mathcal{I}$ *if* $G_1 \cong G_n$, $G_2 \cong G_{n-1}$, *and so on.*

# Further research

- Circle graphs
- Circular-arc graphs
- Intersection graphs in general

# Thank you!



$$\mathrm{Aut}(G) \cong (\mathbb{Z}_2 \rtimes \mathbb{Z}_2) \times \mathbb{Z}_2$$