Large digraphs of given degree and diameter and their properties

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Introduction

Motivation:
- Design of interconnection networks

Degree-diameter problem for graphs and digraphs:
- Determination of the largest number $n(d, k)$ of vertices in a graph (digraph) of a given maximum degree $d$ and diameter $k$

Restricted classes of graphs and digraphs:
- Vertex-transitive, Cayley, bipartite graphs (digraphs), graphs (digraphs) embeddable in a fixed surface...
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Directed case

Upper bounds:

Directed Moore bound $M'(d, k) = n(d, k) \leq 1 + d + d^2 + \ldots + d^k = M'(d, k)$

Moore digraphs: $d = 1 \ldots C_{k+1}$, $k \geq 1; k = 1 \ldots K_{d+1}$ for $d \geq 1$. 
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- Moore digraphs: $d = 1 \ldots C_{k+1}$, $k \geq 1$;
  \[ k = 1 \ldots K_{d+1} \text{ for } d \geq 1. \]
Vertex-transitive case

A digraph is vertex-transitive if for every pair of vertices \( u \) and \( v \) there exists an automorphism of \( \Gamma \) that carries \( u \) to \( v \). 

\( v^t(d,k) \)-the largest order of a vertex-transitive digraph of maximum degree \( d \) and diameter at most \( k \). 

\( v^t(d,k) = M'(d,k) \) is attained only in the trivial cases if \( d = 1 \) or if \( k = 1 \). 

There is no general upper bound on \( v^t(d,k) \) better than \( M'(d,k) - 1 \), for \( d, k \geq 2 \).
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- \( vt(d, 2) = M'(d, 2) - 1 \) for all \( d \geq 2 \) ... (line digraphs of complete digraphs are vertex-transitive)
**Vertex-transitive case**

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- $vt(d, k) = M'(d, k)$ is attained only in the trivial cases if $d = 1$ or if $k = 1$.

- $vt(d, 2) = M'(d, 2) - 1$ for all $d \geq 2$... (line digraphs of complete digraphs are vertex-transitive)

- There is **no** general upper bound on $vt(d, k)$ **better** than $M'(d, k) - 1$, for $d, k \geq 2$. 
Constructions

Faber-Moore-Chen digraphs:
$$v_t(d, k) \geq \frac{(d+1)!}{(d+1-k)!}.$$  

Comellas Fiol digraphs: harder to be extracted, the construction depends on an input digraph of order $n$, degree $d$, and reachability $r$ and two additional numerical parameters $t$ and $l$. For two positive integers $t$ and $l$, the output digraph is a $d$-regular vertex-transitive digraph of diameter at most $(r+t)l-1$ and order $tln$.

Gómez digraphs: a new family of large vertex transitive digraphs, giving the bound
$$n(d, k) \geq \frac{(d+\lceil k-1/2 \rceil)!}{(d-\lceil k+1/2 \rceil)!}$$ for $k \geq 3$ and $d \geq \lceil k+1/2 \rceil$. 

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Lower bounds:

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Faber-Moore-Chen digraphs

For any given $k \geq 2$ and $d \geq k$ their construction gives a family of large vertex-transitive digraphs $\Gamma(d, k)$:

Vertices of $\Gamma(d, k)$ are the different words $x_1x_2...x_k$ of length $k$, forming a $k$-permutation of an alphabet $A$ of $d + 1$ letters.

Adjacencies are given by:

\[
\begin{align*}
  x_1x_2...x_k &\rightarrow \begin{cases} 
    x_2x_3...x_k, & x_{k+1} \neq x_1, x_2, ..., x_k, \\
    x_2x_3...x_k, & x_{k+1} = x_k, \\
    x_1x_2...x_k, & x_{k+1} = x_1.
  \end{cases}
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Order: $(d+1)^k = (d+1)!/(d-k+1)!$, diameter $k$, and $d$-regular.
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\begin{align*}
    x_1x_2\ldots x_k \rightarrow & \left\{ \begin{array}{l}
        x_2x_3x_4\ldots x_{k+1}, \quad x_{k+1} \neq x_1, x_2, \ldots, x_k \\
        x_2x_3x_4\ldots x_kx_1 \\
        x_1x_3x_4\ldots x_kx_2 \\
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Theorem

The automorphism group of the Faber-Moore-Chen digraphs is isomorphic to $S_{d+1}$ acting on $V(\Gamma)$ in a natural way, that is, for any $\sigma \in S_{d+1}$, the assignment $x_1 \ldots x_k \mapsto \sigma(x_1)\sigma(x_2)\ldots\sigma(x_k)$ defines an automorphism of $\Gamma$. 

Theorem

The digraph $\Gamma(d, k)$ is a Cayley digraph if and only if

a) $d = k$

b) $d = k + 1$ and any $k$,

c) $d = q - 1$, where $q$ is a prime power and $k = 2$,

d) $d = q$, where $q$ is a prime power and $k = 3$,

e) $d = 10$ and $k = 4$,

f) $d = 11$ and $k = 5$. 

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- Explicit constructions of large Cayley digraphs of given (general) out-degree and diameter have not been considered in the literature.
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- We characterized those Faber-Moore-Chen digraphs which are Cayley digraphs - with the help of classification of sharply $k$-transitive groups of a given degree.
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- We show that the Cayley digraphs arising from the Faber-Moore-Chen construction can be derived directly from the ‘first principles’: just from the definition of sharply $k$-transitive groups for $k \geq 2$, and not involving any theory of such groups and any prior knowledge of the Faber-Moore-Chen digraphs.
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Preliminaries

Permutation group: A group of permutations $G$ of a set $X$. $G$ is transitive if for any $a, b \in X$, there exists $g \in G$, such that $g(a) = b$. $G$ is $k$-transitive if for any ordered $k$-tuples $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ of distinct elements of $X$ there exists some $g \in G$ such that $g(a_i) = b_i$ for all $i$, $1 \leq i \leq k$. If this element $g$ is unique, then $G$ is sharply $k$-transitive.
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- If this element $g$ is unique, then $G$ is sharply $k$-transitive.
Examples of such Cayley digraphs can be obtained from examples of sharply $k$-transitive groups, which is especially handy in the cases $k = 2$ and $k = 3$. 

**Proposition 1**
Let $G$ be a sharply 2-transitive permutation group on a finite set $S$ with $|S| \geq 2$, so that $|G| = |S|(|S| - 1)$. Then, there exists a Cayley digraph $\Gamma_1$ of the group $G$ of degree $|S| - 1$ and diameter 2.

**Proposition 2**
Let $G$ be a sharply 3-transitive permutation group on a finite set $S$ with $|S| \geq 3$, so that $|G| = |S|(|S| - 1)(|S| - 2)$. Then, there exists a Cayley digraph $\Gamma_2$ of the group $G$ of degree $|S| - 1$ and diameter 3.
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**Proposition 2** Let \( G \) be a sharply 3-transitive permutation group on a finite set \( S \) with \( |S| \geq 3 \), so that \( |G| = |S|(|S| - 1)(|S| - 2) \). Then, there exists a Cayley digraph \( \Gamma_2 \) of the group \( G \) of degree \( |S| - 1 \) and diameter 3.
For diameter 2 the only 2-sharply transitive groups are the groups of linear transformations $\{x \mapsto ax + b : a, b \in F, a \neq 0\}$ of a finite nearfield $F$ of order $q = p^m$, where $p$ is a prime.
For diameter 2 the only 2-sharply transitive groups are the groups of linear transformations \( \{x \mapsto ax + b : a, b \in F, a \neq 0\} \) of a finite nearfield \( F \) of order \( q = p^m \), where \( p \) is a prime.

By a Theorem of Zassenhaus, there are two infinite families of sharply 3-transitive groups: \( PGL_2(q) \), where \( q = p^m \) and \( M(q) \), where \( q = 2^m \) if \( m \) is even.
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There are just two sharply \( k \)-transitive groups: the Mathieu group \( M_{11} \) for \( k = 4 \) and the Mathieu group \( M_{12} \) for \( k = 5 \).
For diameter 2 the only 2-sharply transitive groups are the groups of linear transformations \( \{ x \mapsto ax + b : a, b \in F, a \neq 0 \} \) of a finite nearfield \( F \) of order \( q = p^m \), where \( p \) is a prime.

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Digraphs: from Proposition 1 - $\Gamma_1$, from Proposition 2 - $\Gamma_2$
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\( \Gamma(d, 2) \)-FMCH digraphs for diameter 2

\( \Gamma(d, 3) \)-FMCH digraphs for diameter 3
Digraphs: from Proposition 1 - $\Gamma_1$, from Proposition 2 - $\Gamma_2$

$\Gamma(d,2)$-FMCH digraphs for diameter 2
$\Gamma(d,3)$-FMCH digraphs for diameter 3

**Theorem**

*The digraph $\Gamma_1$ is isomorphic to the digraph $\Gamma(d,2)$, and the digraph $\Gamma_2$ is isomorphic to the digraph $\Gamma(d,3)$.***
Construction of Comellas and Fiol

Introduction

Construction of Comellas and Fiol

Let \( l, t \) be positive integers.

Vertices of \( \text{CF}(\Gamma, l, t) \) are \((j | p_0 \ldots p_{l-1})\) with \( j \in \mathbb{Z}_{lt} \) and \( p_i \in V = \mathbb{Z}_n \).

Adjacencies are given by:

\[
(j | p_0 \ldots p_{j-1}u \ p_{j+1} \ldots p_{l-1}) \rightarrow (j+1 | p_0 \ldots p_{j-1}v \ p_{j+1} \ldots p_{l-1}),
\]

where \((u, v) \in A\).

Order: \( tl \) and diameter at most \((r + t) - 1\).
Construction of Comellas and Fiol

- joint work with Ľ. Staneková
Construction of Comellas and Fiol

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- \( \Gamma \) - an input \( r \)-reachable digraph of out-degree \( d \), with vertex set \( V \) and arc set \( A \).
Construction of Comellas and Fiol

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- $\Gamma$ - an input $r$-reachable digraph of out-degree $d$, with vertex set $V$ and arc set $A$.
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Construction of Comellas and Fiol

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- $\Gamma$ - an input $r$-reachable digraph of out-degree $d$, with vertex set $V$ and arc set $A$.
- Let $l, t$ be positive integers.
- Vertices of $CF(\Gamma, l, t)$ are $(j | p_0 p_1 \ldots p_{l-1})$ with $j \in \mathbb{Z}_{lt}$ and $p_i \in V = \mathbb{Z}_n$. 

\[ \text{Adjacencies are given by:} \]
\[ (j | p_0 p_1 \ldots p_{j-1} u p_{j+1} \ldots p_{l-1}) \rightarrow (j+1 | p_0 p_1 \ldots p_{j-1} v p_{j+1} \ldots p_{l-1}) , \]
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\[ \text{Order: } tl \text{ and diameter at most } l(r + t) - 1. \]
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- joint work with Ž. Staneková
- $\Gamma$ - an input $r$-reachable digraph of out-degree $d$, with vertex set $V$ and arc set $A$.
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Introduction

Results
Theorem

If the input digraph $\Gamma$ is $r$, $r + 1$, $r + 2$-alternately reachable and vertex-transitive, then the order of the automorphism group of the Comellas-Fiol digraph $CF(\Gamma, l, 1)$ is $|\text{Aut}(CF(\Gamma, l, 1))| = l.|\text{Aut}(\Gamma)|^l$. Consequently, $\text{Aut}(CF(\Gamma, l, 1)) \cong [\text{Aut}(\Gamma)]^l \rtimes \mathbb{Z}_l$. 
We have determined the full automorphism group of Comellas-Fiol digraphs, when $t = 1$ and the input digraphs are \{r, r + 1, r + 2\}-alternately reachable.
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Computer experiments: the automorphism group $H$ of the output digraph may be bigger than the group from our main result: $|H| > lt|G|'$.
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**Example:** Consider the digraph \( \Gamma_1 \) with the vertex set \( V(\Gamma_1) = \mathbb{Z}_6 \) and the dart set \((i, i + 1), (i, i + 2), i \in \mathbb{Z}_6\). Let the automorphism group of the digraph \( \Gamma_1 \) and corresponding output digraph \( CF(\Gamma_1, l, 2) \) for the parameters \( l = t = 2 \) be \( G_1 \) and \( H_1 \), respectively. With the help of a computer we have \( |G_1| = 6 \) and \( |H_1| = 576 > 2.2|G_1|^2 = 144 \).
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- Faber-Moore-Chen digraphs: The full automorphism group of the digraphs.
- Comellas-Fiol digraphs: Automorphism group of the digraphs for special input parameters.
- Open problem: Automorphism group of Gómez’s digraphs.
Thank you for your attention.