Regular embeddings of complete bipartite graphs with multiple edges

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Maps on orientable surfaces

Topological oriented map a 2-cell embedding of a connected (multi)graph into a closed oriented surface.

Combinatorial map a triple $\mathcal{M} = (D; R, L)$, where $D \neq \varnothing$ finite, $R, L \in Sym(D)$, $L^2 = 1$, and $\mathrm{Mon}(\mathcal{M}) = \langle R, L \rangle$ acts transitively on D.

Homomorphism $\mathcal{M}_1 \to \mathcal{M}_2$ is a mapping $\sigma\colon D_1 \to D_2$ such that $\sigma R_1 = R_2 \sigma$ and $\sigma L_1 = L_2 \sigma$.

Automorphism a homomorphism $\mathcal{M} \to \mathcal{M}$.

Map symmetries

Semiregularity the automorphism group $\mathrm{Aut}\,(\mathcal{M})$ acts semi-regularly on D.

Regular map $\mathrm{Aut}\,(\mathcal{M})$ acts regularly on D, in which case, $\mathrm{Aut}\,(\mathcal{M}) \cong \mathrm{Mon}\,(\mathcal{M})$.

Reflexibility $\mathcal{M}$ is reflexible if $\mathcal{M}$ admits orientation-reversing automorphisms, otherwise, it is chiral.

Algebraic map

Regular maps=Algebraic maps

Each regular map $\mathcal{M}$ with $G = \mathrm{Aut}\,(\mathcal{M})$ can be identified with a triple $(G; r, l)$, where $G = \langle\, r, l\,\rangle$, r stabilizes a vertex, and l is an involution stabilizing an incident edge.

Two regular maps $(G_1; r_1, l_1)$ and $(G_2; r_2, l_2)$ are isomorphic, if the assignment $r_1 \mapsto r_2$, $l_1 \mapsto l_2$ extends to an isomorphism $G_1 \cong G_2$.

Regular bipartite maps

- Each regular bipartite map $\mathcal{M} = (G; r, l)$ has an index-2 subgroup $H = \mathrm{Aut}_0(\mathcal{M}) = \langle x, y \rangle$ of color-preserving automorphisms, where $x = r$, $y = r^l$.
- $G = H \rtimes \langle l \rangle$, where l induces $\tau \in \mathrm{Aut}(H)$ which transposes x and y.

Tranpositive triples

Definition

A triple (H, x, y) will be called a transpositive triple of degree d if

1. H is a finite group generated by x and y,
2. H admits an automorphism τ transposing x and y,
3. $o(x) = o(y) = d$.

Two transpositive triples (H_i, x_i, y_i) $(i = 1, 2)$ are equivalent if the assignment $x_1 \mapsto x_2$, $y_1 \mapsto y_2$ extends to an isomorphism from H_1 onto H_2.

Regular bipartite maps vs transpositive triples

Theorem (Jones, Nedela, Škoviera, 2007)

The isomorphism classes of regular bipartite maps $\mathcal{M}$ of valency d with $H = \mathrm{Aut}_0(\mathcal{M})$ are in 1-1 correspondence with the equivalence classes of transpositive triples (H, x, y) of degree d. In particular,

- *$\mathcal{M}$ is reflexible iff $x \mapsto x^{-1}$, $y \mapsto y^{-1}$ extends to an automorphism of H;*
- *If $K = \langle x \rangle \cap \langle y \rangle \neq 1$, then $\mathcal{M} = (G; x, y)$ projects onto $\bar{\mathcal{M}} = (G/K; xK, yK)$ with a simple underlying graph Γ, and the underlying graph of $\mathcal{M}$ is $\Gamma^{(m)}$, where $m = |K|$.*

Regular bipartite maps $\mathcal{M}$ with $\mathrm{Aut}_0(\mathcal{M}) \cong \mathrm{Alt}_4$

Example

Let $x = (123), y = (124)$. Then

$$H = \mathrm{Alt}_4 = \langle\, x, y \mid x^3 = y^3 = (xy)^2 = (x^{-1}y)^3 = 1 \,\rangle,$$
$$H = \mathrm{Alt}_4 = \langle\, x^{-1}, y \mid x^{-3} = y^3 = (x^{-1}y)^3 = (xy)^2 = 1 \,\rangle.$$

- (H, x, y) and (H, x^{-1}, y) are non-equivalent transpositive triples, corresponding to the spherical cube and the toroidal cube respectively.
- By identifying $\mathrm{Aut}\,(H) = \mathrm{Sym}_4$, then $\tau_1 = (34)$ transposes x and y, $\tau_2 = (12)(34)$ transposes x^{-1} and y.

Trivial or non-trivial?

Proposition (trivial)

Up to isomorphism, there are exactly 2 regular bipartite maps $\mathcal{M}$ with $\mathrm{Aut}_0(\mathcal{M}) \cong \mathrm{Alt}_4$.

Open questions

- How many regular bipartite maps $\mathcal{M}$ with $\mathrm{Aut}_0(\mathcal{M})$ isomorphic to the n-dimensional affine groups $\mathrm{Aff}(n, q)$? (note that $\mathrm{Alt}_4 \cong \mathrm{Aff}(1, 4)$)
- How many regular bipartite maps $\mathcal{M}$ with $\mathrm{Aut}_0(\mathcal{M})$ isomorphic to the alternating groups Alt_n?

Regular embeddings of the cube with multiple edges

Theorem

- *The isomorphism classes of regular embeddings of $Q_3^{(m)}$ which project onto the spherical regular embedding of the cube Q_3 are in one-to-one correspondence with the solutions (r, s) in $\mathbb{Z}_m$ of the system of congruences*

$$r^2 \equiv 1 \pmod{m},$$
$$s(r-1) \equiv 0 \pmod{m},$$
$$4(r+1) + 6s \equiv 0 \pmod{m}.$$

Theorem (Continued)

- *The isomorphism classes of regular embeddings of $Q_3^{(m)}$ which project onto the toroidal regular embedding of the cube Q_3 are in one-to-one correspondence with the solutions (r, s) in $\mathbb{Z}_m$ of the system of congruences:*

$$r^2 \equiv 1 \pmod{m},$$
$$s(r-1) \equiv 0 \pmod{m},$$
$$2(2+s) \equiv 0 \pmod{m},$$
$$2(r-1) \equiv 0 \pmod{m}.$$

Number of embeddings of $Q_3^{(m)}$

m	e_0	μ_0	μ_1	μ
$3 \mid m$	$e_0 = 0$	2^{n-1}	1	$2^{n-1}+1$
	$e_0 = 1$	2^{n}	2	$2^{n}+2$
	$e_0 = 2$	2^{n+1}	4	$2^{n+1}+4$
	$e_0 \geq 3$	2^{n+2}	4	$2^{n+2}+4$
$3 \nmid m$	$e_0 = 0$	2^{n}	1	$2^{n}+1$
	$e_0 = 1$	2^{n+1}	2	$2^{n+1}+2$
	$e_0 = 2$	2^{n+2}	4	$2^{n+2}+4$
	$e_0 \geq 3$	2^{n+3}	4	$2^{n+3}+4$

μ: number of regular embeddings of $Q_3^{(m)}$, where $m = 2^{e_0} p_1^{e_1} \cdots {p_n}^{e_n}$.

Isobicyclic triples

A transpositive triple (H, x, y) of degree n is called an n-isobicyclic triple if $H = \langle x \rangle \langle y \rangle$ and $\langle x \rangle \cap \langle y \rangle = 1$.

Theorem (Jones-Nedela-Škoviera, 2007)

The isomorphism classes of regular embeddings of $K_{n,n}$ are in 1-1 correspondence with the equivalence classes of n-isobicyclic triples.

The standard embedding

Example

$K_{n,n}$ admits at least one regular embedding - the standard embedding corresponding to the n-isobicyclic triple $(\mathbb{Z}_n \times \mathbb{Z}_n, (1,0), (0,1))$.

Theorem (Jones-Nedela-Škoviera, 2008)

$K_{n,n}$ admits a unique regular embedding iff $\gcd(n, \varphi(n)){=}1$.

Decomposing regular embeddings of $K_{n,n}$

Theorem (Jones, 2010)

Each n-isobicyclic triple is equivalent to a semidirect product of a t-isobicyclic triple (T, x_T, y_T) by an s-isobicyclic triple (S, x_S, y_S), where

1. $n = st$, $\gcd(s,t) = 1$,
2. (S, x_S, y_S) *is a Cartesian product of $p_i^{e_i}$-isobicyclic triples for $i = 1, \cdots, k$, where $s = p_1^{e_1} \cdots p_k^{e_k}$ for distinct primes p_i,*
3. (T, x_T, y_T) *corresponds to the standard embedding of $K_{t,t}$*

Solution

Classification of regular embeddings $\mathcal{M}$ of K_{p^e,p^e}, where p is a prime:

- Jones-Nedela-Škoviera (2008) for $p > 2$ where $\mathrm{Aut}_0(\mathcal{M})$ must be metacyclic;
- Du-Jones-Kwak-Nedela-Škoviera (2008) for $p = 2$ and $\mathrm{Aut}_0(\mathcal{M})$ is metacyclic (2008), and for $p = 2$ and $\mathrm{Aut}_0(\mathcal{M})$ is non-metacyclic (2010).

Extended problems

Problem

- *Classify regular bipartite maps with nilpotent color-preserving automorphism groups: answered for nilpotency class $c \le 2$ (Hu, Nedela and W.)*
- *Classify regular embeddings of complete bipartite graphs with multiple edges: partially answered, see below.*

Regular embeddings of $K_{n,n}^{(m)}$ over the standard embedding of $K_{n,n}$

Theorem

- *The isomorphism classes of regular embeddings $\mathcal{M}$ of $K_{n,n}^{(m)}$ which project onto the standard embedding of the complete bipartite graph $K_{n,n}$ are in 1-1 correspondence with the solutions (r, s) in $\mathbb{Z}_m$ of the system of congruences:*

$$r^2 \equiv 1 \pmod{m},$$
$$s(r+1) \equiv 0 \pmod{m},$$
$$ns \equiv 0 \pmod{m}.$$

Theorem Continued

- The group $\mathrm{Aut}_0(\mathcal{M})$ has a presentation

$$\langle\, x, y \mid x^{nm} = 1, y^n = x^{nr}, [x, y] = x^{ns} \,\rangle\,.$$

- The map $\mathcal{M}$ is reflexible iff $2s \equiv 0 \pmod{m}$.

Regular embeddings of $K_{4,4}$

Proposition (Du-Jones-Kwak-Nedela-Škoviera, 2010)

The graph $K_{4,4}$ admits exactly 2 regular embeddings: a standard embedding $\mathcal{B}_3$ into the triple torus, and a toroidal regular embedding $\mathcal{B}_1$ with $\mathrm{Aut}_0(\mathcal{B}_1)$ non-metacylcic.

Regular embeddings of $K_{4,4}^{(m)}$ over $\mathcal{B}_1$

Theorem

The isomorphism classes of regular embeddings of $K_{4,4}^{(m)}$ which project onto the toroidal regular embedding $\mathcal{B}_1$ of $K_{4,4}$ are in 1-1 correspondence with the solutions (s,t) in $\mathbb{Z}_m$ of the system of congruences:

$$2s \equiv 0 \pmod{m},$$
$$2t(1-t) \equiv 0 \pmod{m}.$$

Number of regular embeddings of $K_{4,4}^{(m)}$

e_0	ν_1	ν_3	ν
$e_0 = 0$	2^n	2^n	2^{n+1}
$e_0 = 1$	2^{n+2}	2^{n+1}	$3 \cdot 2^{n+1}$
$e_0 = 2$	2^{n+3}	$3 \cdot 2^{n+1}$	$7 \cdot 2^{n+1}$
$e_0 \geq 3$	2^{n+3}	$3 \cdot 2^{n+2}$	$5 \cdot 2^{n+2}$

ν: total number of regular embeddings of $K_{4,4}^{(m)}$, where $m = 2^{e_0} p_1^{e_1} \cdots p_n^{e_n}$.

What else has been done?

We have classified:

- regular embeddings of $K^{(m)}_{p^e,p^e}$ which project onto a regular embedding $\mathcal{M}$ of K_{p^e,p^e} with $\mathrm{Aut}_0(\mathcal{M})$ metacyclic, for both $p > 2$ and $p = 2$, and for arbitrary m.
- regular embeddings of $K^{(m)}_{2^e,2^e}$ which project onto a regular embedding $\mathcal{M}$ of $K_{2^e,2^e}$ with $\mathrm{Aut}_0(\mathcal{M})$ non-metacyclic for $m = 2^f$ and $e \geq 3$.

Still open

Problem

Classification of regular embeddings of $K_{n,n}^{(m)}$ for arbitrary n and m.

This is equivalent to

Problem

Classification of transpositive triples (H, x, y) of degree nm such that

$$H = \langle x \rangle \langle y \rangle \quad and \quad \langle x \rangle \cap \langle y \rangle \cong \mathbb{Z}_m.$$

Thank you for your attention!