Counting spanning trees

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Spanning Trees.

Spanning tree

A *spanning tree* $T$ of a connected, undirected graph $G$ is a tree composed of all the vertices and some (or perhaps all) of the edges of $G$. In other words, a spanning tree of $G$ is a selection of edges of $G$ that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are allowed. On the other hand, every bridge of $G$ must belong to $T$. A spanning tree of a connected graph $G$ can also be defined as a maximal set of edges of $G$ that contains no cycle, or as a minimal set of edges that connect all vertices.
Counting spanning trees

The number $t(G)$ of spanning trees of a connected graph is a well-studied invariant. In some cases, it is easy to calculate $t(G)$ directly. For example, if $G$ is itself a tree, then $t(G) = 1$, while if $G$ is the cycle graph $C_n$ with $n$ vertices, then $t(G) = n$. For any graph $G$, the number $t(G)$ can be calculated using Kirchhoff’s matrix-tree theorem.

Here are some known results concerning counting spanning trees of graphs.

1. Complete graph $K_n$: $t(K_n) = n^{n-2}$ (Cayley’s formula),

2. Complete bipartite graph $K_{n,m}$: $t(K_{n,m}) = m^{n-1} n^{m-1}$,

3. $n$-dimensional cube graph $Q_n$: $t(Q_n) = 2^{2^n-n-1} \prod_{k=2}^{n} k^{\binom{n}{k}}$. 

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Kirchhoff Matrix-Tree Theorem
The celebrated Kirchhoff Matrix-Tree Theorem is the following statement.

**Theorem (Kirchhoff (1847))**

All cofactors of Laplacian matrix $L(G)$ are equal to $t(G)$.

More convenient form of this result were obtained by A. K. Kel’mans and V. M. Chelnokov.

**Theorem (Kel’mans, Chelnokov (1974))**

Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ denote the eigenvalues of the Laplace matrix $L(G)$ of a $n$ point graph $G$. Then

$$t(G) = \frac{1}{n} \prod_{k=2}^{n} \lambda_k.$$
The Temperley’s formula
One more convenient way to count spanning trees.

Theorem (Temperley, H. N. V. (1964))

The number of spanning trees of a $n$ point graph $G$ is given by the formula

$$t(G) = \det(L(G) + \frac{1}{n^2} J),$$

where $J$ is $n \times n$ matrix all of whose elements are unity.

Now we will present an uniform proof for all the three previous theorems.
Spanning Trees

Proof of Kirchhoff, Kal’mans - Chelnokov and Temperley theorems

Proof.

Let $L = L(G)$ be the Laplacian matrix of $G$ with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The $(i, j)$-cofactor of a matrix $M$ is by definition $(-1)^{i+j} \det M(i, j)$, where $M(i, j)$ is the matrix obtained from $M$ by deleting row $i$ and column $j$.

Let $l_{xy}$ be the $(x, y)$-cofactor of $L$. Note that $l_{xy}$ does not depend on an ordering of the vertices of $G$.

We set $N = t(G)$ and show that

$$N = l_{xy} = \det(L + \frac{1}{n^2}J) = \frac{1}{n} \lambda_2 \ldots \lambda_n$$

for any $x, y \in V(G)$. 
Let $L^S$, for $S \subset V(G)$, denote the matrix obtained from $L$ by deleting the rows and columns indexed by $S$, so that $l_{xx} = \det L\{x\}$. The equality $N = l_{xx}$ follows by induction on $n$, and for fixed $n > 1$ on the number of edges incident with $x$. Indeed, if $n = 1$ then $l_{xx} = 1$. Otherwise, if $x$ has degree 0, then $l_{xx} = 0$ since $L\{x\}$ has zero row sums. Now, if $xy$ is an edge, then deleting this edge from $G$ decreases $l_{xx}$ by $\det L\{x,y\}$, which by induction is the number of spanning trees of $G$ with edge $xy$ collapsing to a point, which is the number of spanning trees containing the edge $xy$. This shows $N = l_{xx}$. Since the sum of the columns of $L$ is zero, so that one column is minus the sum of the other columns, we have $l_{xx} = l_{xy}$ for any $x, y$. 
Now we consider the Laplacian polynomial
\[ \mu(G, t) = \det(tI - L) = t \prod_{i=2}^{n} (t - \lambda_i) \] for graph \( G \). Then
\[ (-1)^{n-1} \lambda_2 \ldots \lambda_n \] is the coefficient of \( t \), that is,
\[ \frac{d}{dt} \det(tI - L) \bigg|_{t=0}. \]
We note that
\[ \frac{d}{dt} \det(tI - L) = \sum_{x} \det(tI - L\{x\}). \]

Putting \( t = 0 \) we obtain
\[ \lambda_2 \ldots \lambda_n = \sum_{x} l_{xx} = nN. \]

Finally, the eigenvalues of \( L + \frac{1}{n^2} J \) are \( \frac{1}{n} \) and \( \lambda_2, \ldots, \lambda_n \), so
\[ \det(L + \frac{1}{n^2} J) = \frac{1}{n} \lambda_2 \ldots \lambda_n. \]
A generalization of the Matrix-Tree-Theorem was obtained by Kelmans (1967) who gave a combinatorial interpretation to all the coefficients of \( \mu(G, x) \) in terms of the numbers of certain subforests of the graph. This result has been obtained even in greater generality (for weighted graphs) by Fiedler and Sedláček.

**Theorem (Kel’mans (1967))**

If

\[
\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x
\]

then

\[
c_i = \sum_{S \subset V, |S| = n-i} t(G_S),
\]

where \( t(H) \) is the number of spanning trees of \( H \), and \( G_S \) is obtained from \( G \) by identifying all vertices of \( S \) to a single one.
From the last theorem we can derive useful corollary.

**Corollary**

*The degree of Laplacian polynomial* $\mu(G, x)$ *is equal to* $n = |V(G)|$. *Its coefficients* $c_1$ *and* $c_{n−1}$ *are given by the formulas* $c_1 = 2|E(G)|$ *and* $c_{n−1} = |V(G)| \cdot t(G)$. 

Hence, the number of vertices $|V(G)|$, number of edges $|E(G)|$ and the number of spanning trees $t(G)$ are uniquely defined by the Laplacian polynomial.
Some recursive formulas for $t(G)$

Now we give a few recursive formulas for the number of spanning trees employed in graph theory by W. Feussner (1904) and J. W. Moon (1970).

Denote by $G - e$ the graph obtained by removing edge $e$ from the graph $G$. Let $G\backslash e$ be the graph obtained from graph $G$ by contracting edge $e$. In other words, $G\backslash e$ is obtained by deleting edge $e$ and identifying its ends. Then the following formula takes a place.

$$t(G) = t(G - e) + t(G\backslash e).$$

**Proof.** We note that the set of spanning trees of a given graph $G$ decomposed in two disjoint sets. First set consist of tree containing selected edge $e \in E(G)$ and second set consist of trees that do not contain $e$. The number of spanning trees that contains $e$ is exactly $t(G\backslash e)$ because each of them corresponds to a spanning tree of $G\backslash e$. The number of spanning tree that do not contain $e$ is $t(G - e)$, since each of them is also a spanning tree of $G - e$ and vice versa.
Denote by $G_{s,e}$ the graph resulting from subdivision of an edge $e$ of a graph $G$. Then

$$t(G_{s,e}) = t(G\setminus e) + 2t(G - e) = t(G) + t(G - e).$$

**Proof.** Again, the number of spanning trees of $X$ that contains $e$ is $t(G\setminus e)$. All of them also the spanning trees of $t(G_{s,e})$. If a spanning tree of $G$ do not contains $e$ then it can be extended to be a spanning tree of $t(G_{s,e})$ in two different ways. Hence, $t(G_{s,e}) = t(G\setminus e) + 2t(G - e)$. By the previous statement $t(G\setminus e) + 2t(G - e) = t(G) + t(G - e)$. 
Let \( G_{p,e} \) denotes the results of adding an edge in parallel an edge \( e \) of a graph \( G \). Then

\[
t(G_{p,e}) = t(G) + t(G \setminus e).
\]

**Proof.** Distinguishing spanning trees that contain an edge \( e \) and that are not we have

\[
t(G_{p,e}) = t(G - e) + 2t(G \setminus e) = t(G) + t(G \setminus e).
\]
Let \( G_1 \) and \( G_2 \) are the graphs with exactly one vertex in common. Then

\[
t(G_1 \cup G_2) = t(G_1) \cdot t(G_2),
\]

where \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \).

**Proof.** Let \( T \) be a spanning tree of \( G_1 \cup G_2 \). Then \( T_1 = T \cap G_1 \) and \( T_2 = T \cap G_2 \) are spanning trees for \( G_1 \) and \( G_2 \) respectively. Moreover \( G_1 \cap G_2 = \{v\} \), where \( v \) is the common vertex of \( G_1 \) and \( G_2 \). Conversely, let \( T_1 \) and \( T_2 \) are respective spanning trees for \( G_1 \) and \( G_2 \). Then \( T_1 \cap T_2 = \{v\} \) and \( T = T_1 \cup T_2 \) is a spanning tree of \( G_1 \cup G_2 \).
The Chebyshev polynomial of the first kind is defined by the formula

\[ T_n(x) = \cos(n \arccos x). \]

Equivalently,

\[ T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2}. \]

Also, \( T_n(x) \) satisfies the recursive relation

\[ T_0(x) = 1, \ T_1(x) = x, \ T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x), \ n \geq 2. \]
The Chebyshev polynomial of the second kind is defined by the formula

\[ U_n(x) = \frac{\sin((n + 1) \arccos x)}{\sin(\arccos x)}. \]

Equivalently,

\[ U_n(x) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}}. \]

Also, \( U_n(x) \) satisfies the recursive relation

\[ U_0(x) = 1, \quad U_1(x) = 2x, \quad U_n(x) = 2x \cdot U_{n-1}(x) - U_{n-2}(x), \quad n \geq 2. \]

We have \( U_n(\cos \frac{k\pi}{n+1}) = 0, \quad k = 1, 2, \ldots, n. \) Hence

\[ U_n(x) = 2^n \prod_{k=1}^{n} (x - \cos \frac{k\pi}{n+1}). \]
Since

\[ U_n(x) = (-1)^n U_n(-x) = 2^n \prod_{k=1}^{n} (x + \cos \frac{k\pi}{n+1}) \]

we obtain

\[ U_n^2(x) = \prod_{k=1}^{n} (4x^2 - 4 \cos^2 \frac{k\pi}{n+1}). \]

Polynomials \( T_n(x) \) and \( U_{n-1}(x) \) are related by the following identity

\[ T_n^2(x) + (x^2 - 1) U_{n-1}^2(x) = 1. \]
Consider $n \times n$ matrix

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 2x & -1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 2x & -1 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 2x & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2x & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2x \end{pmatrix}.$$ 

Then $\det A_n(x) = U_n(x)$. 