Algorithmic aspects of Regular Graph Covers

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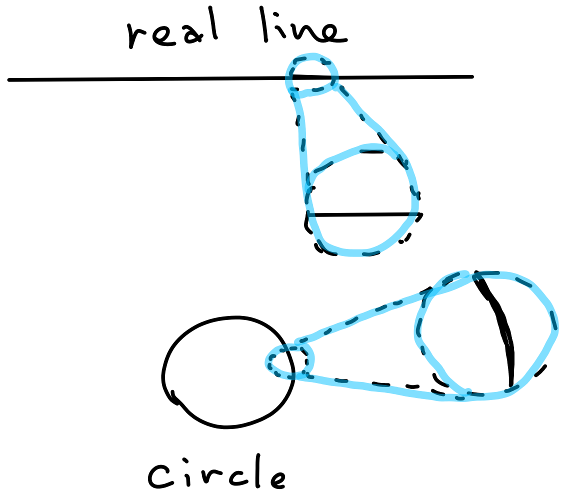
Big picture of this talk:

Revolutions in mathematics are major shifts of focus.

→ Shift to the discrete math and its relation to the continuous one.

→ Focus on algorithmic aspects:
„How hard is to construct?"
We study this for regular graph covers.

What is covering? Why important?



Globally:

Very different surfaces

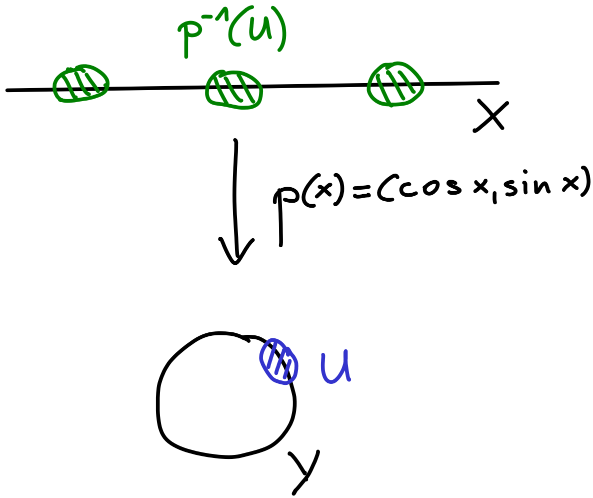
Locally:

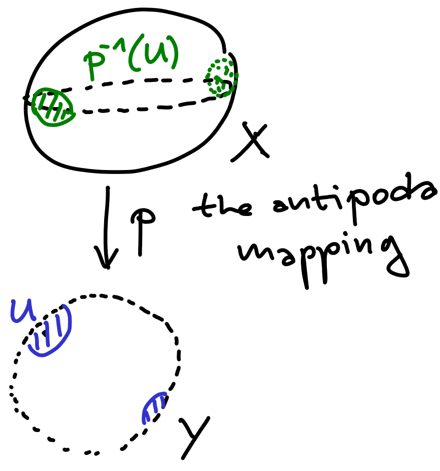
They look the same!

They share their local properties.

Covering formalizes this topological notion:

X covers Y if there exists a mapping $p: X \to Y$ called a covering projection.



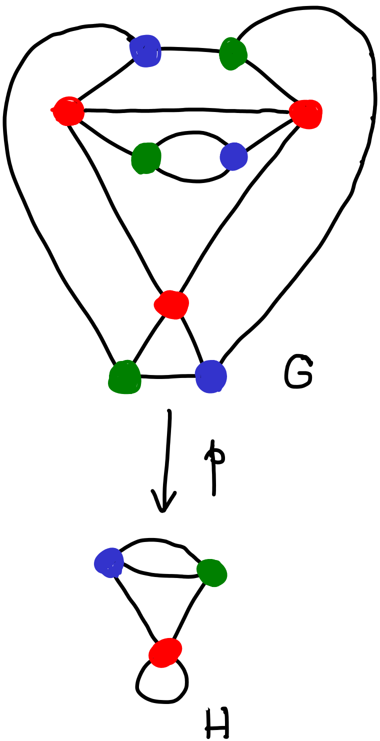


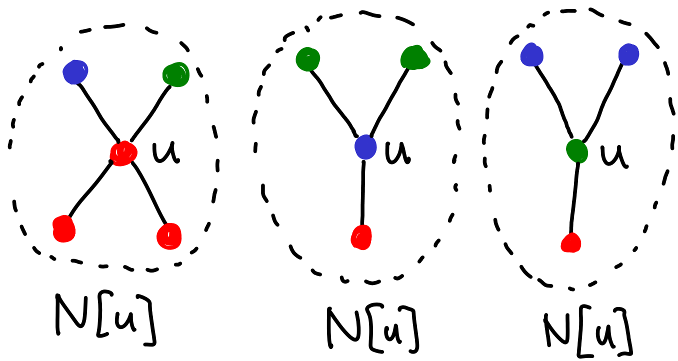
What does it mean for graphs?

As a natural topology, we use the neighborhoods.

$\forall u \in V(G)$

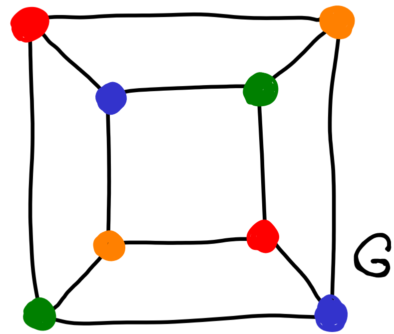
$$p \restriction N[u] \xrightarrow{\text{bij.}} N[p(u)]$$





How I got to study covering?

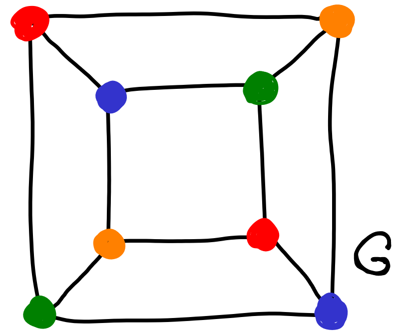
„Given a cubic planar graph, how hard is to test whether there ∃ 4-coloring such that each vertex is adjacent to three distinct colors."

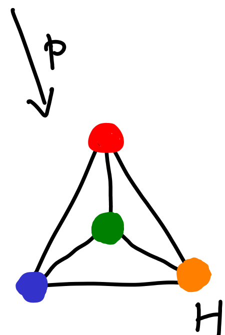


Q: What problem is this?

How I got to study covering?

„Given a cubic planar graph, how hard is to test whether there $\exists$ 4-coloring such that each vertex is adjacent to three distinct colors."

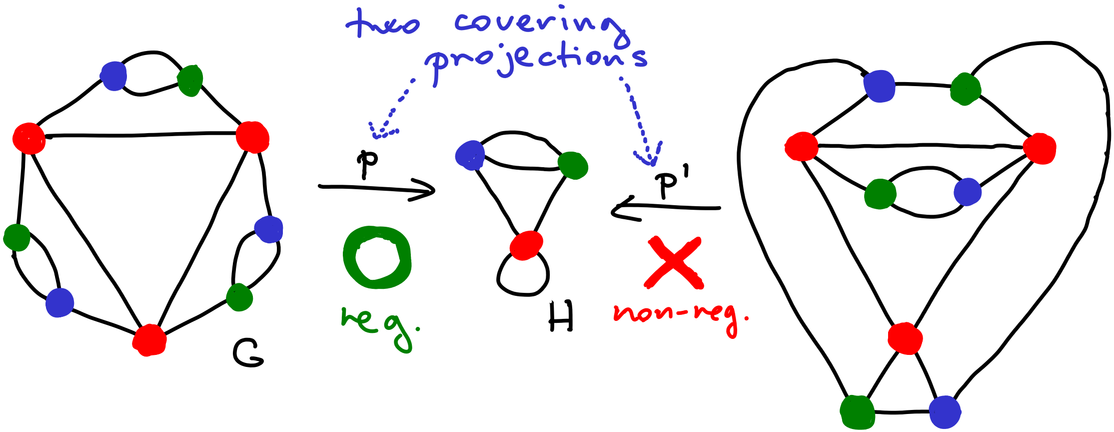




This problem is testing whether an input planar graph covers K_4.

Bílka, Jirásek, K., Tancer, Volec (2011)
NP-hard for several fixed graphs H.

What is regular covering?

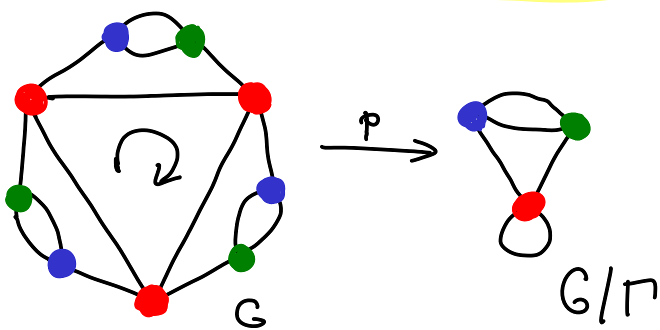


- G covers H ⇒ they look locally the same
- Regularity means that the covering is much more symmetric.

Def.: G regularly covers H, iff there exist a semiregular action $\Gamma \leq \mathrm{Aut}(G)$, such that $G/\Gamma \cong H$.

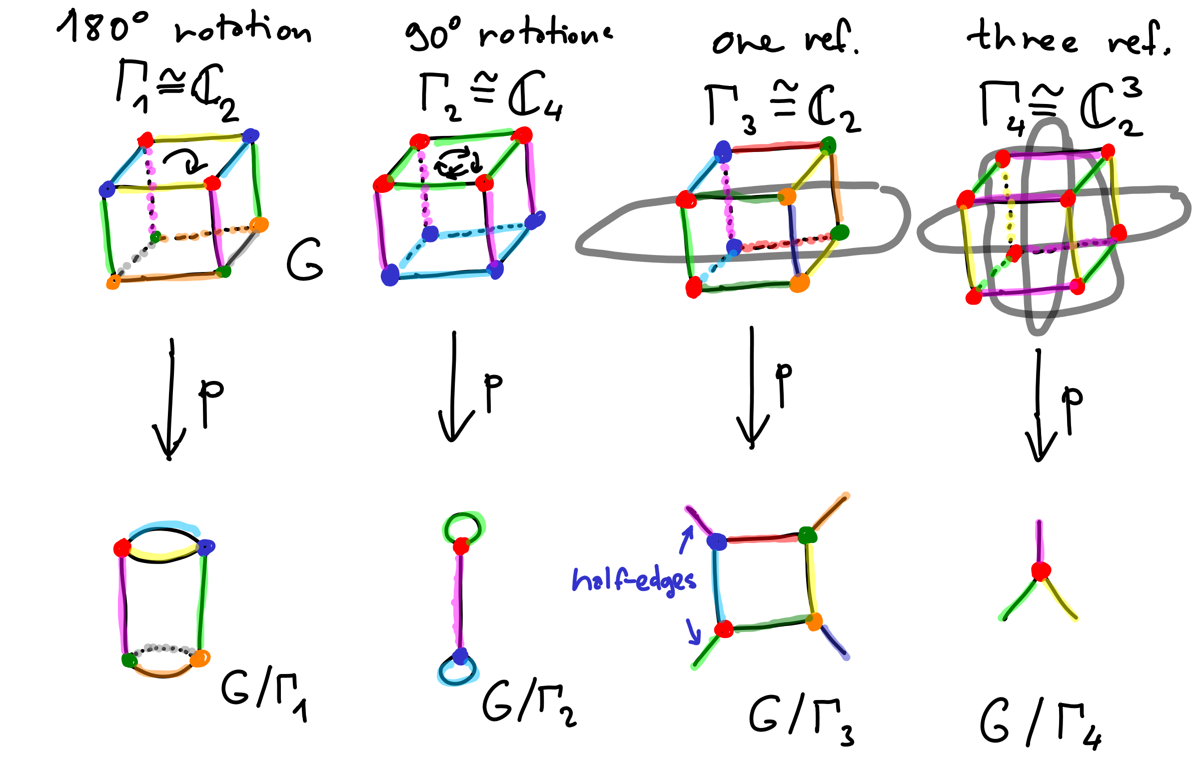
- G/Γ is constructed according to vertex- and edge-orbits of Γ.

$\Gamma = C_3$ 120° rotations

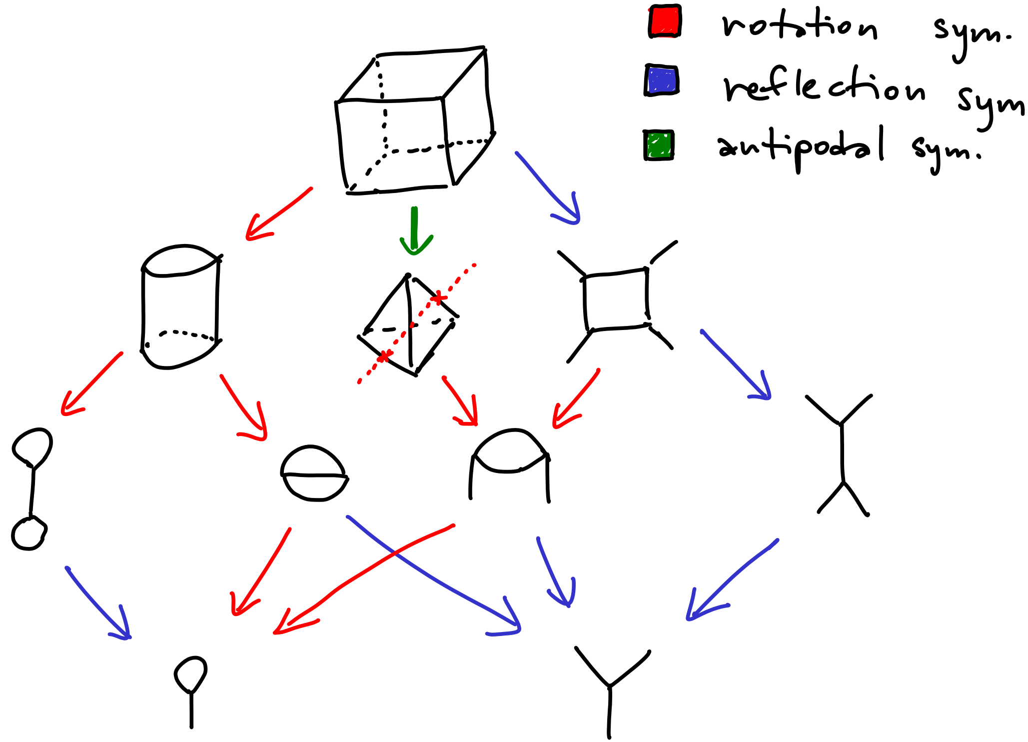


A geometrical example:

$\mathrm{Aut}(\square) = \mathbb{C}_2 \times \mathbb{S}_4$



All quotients of the cube.



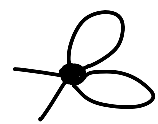
We study the decision problem REGULARCOVER:

Input: Graphs G and H.

Question: Does G regularly cover H?

Two special cases:

- If $|G| = |H|$, then we get the Graph Isomorphism Problem.
- If $|H| = 1$, then it is Cayley Graph Testing.



Closely related to computing $\mathrm{Aut}(G)$.

We decided to start studying of REGULARCOVER for planar inputs G.

1. Negami's Theorem is one of the oldest results of TGT.
2. Automorphism groups of planar graphs behave nicely. Or do they?

Many results independent of planarity.

Common theme: Everything more and more difficult than expected.

We decided to start studying of REGULARCOVER for planar inputs G.

1. Negami's Theorem is one of the oldest results of TGT.
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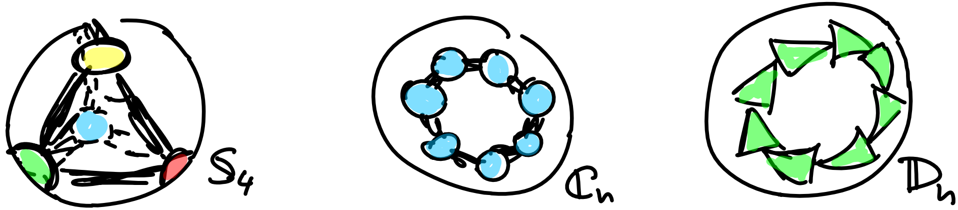
Common theme: Everything more and more difficult than expected.

„It will be very easy to solve this problem." R. Nedela, ATCAGC 2012

Indeed, yes, if G is 3-connected!

Then $\mathrm{Aut}(G)$ is a spherical group.

- $\mathbb{S}_4$, $\mathbb{C}_2 \times \mathbb{S}_4$ or $\mathbb{C}_2 \times \mathbb{A}_5$ or its subgroup.
- Group $\mathbb{C}_n$, $\mathbb{D}_n$ for some n.



Since $k = |\Gamma| = |G|/|H|$, we test all subgroups Γ of $\mathrm{Aut}(G)$ of order k.

For each Γ, compute G/Γ and test $G/\Gamma \cong H$.

But what about 1- and 2-connectivity?

- Negami proved that regular covering projections behave nicely with respect to minimal parts of G called atoms.

Our approach:

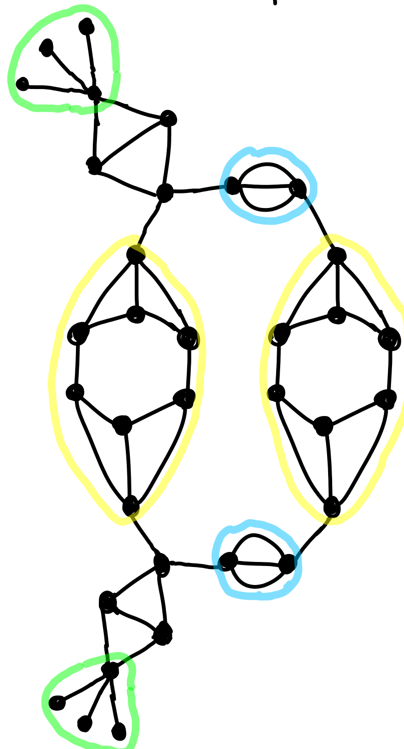
3-connected

called primitive

$$
\begin{tikzcd}
 & G_0 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_0"] & G_1 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_1"] & G_2 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_2"] & \cdots \arrow[r, "\text{red.}"] & G_r \arrow[d, "\Gamma_r"] \\
H \arrow[r, phantom, "\overset{?}{\cong}"] & H_0 & H_1 \arrow[l, "\text{exp.}"'] & H_2 \arrow[l, "\text{exp.}"'] & \cdots \arrow[l, "\text{exp.}"'] & H_r \arrow[l, "\text{exp.}"']
\end{tikzcd}
$$

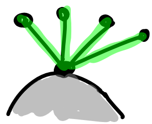
So what are these atoms?

Example:

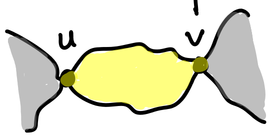


(x) Block atoms are pendant blocks or pendant stars.

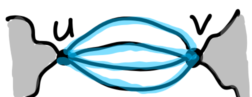
or



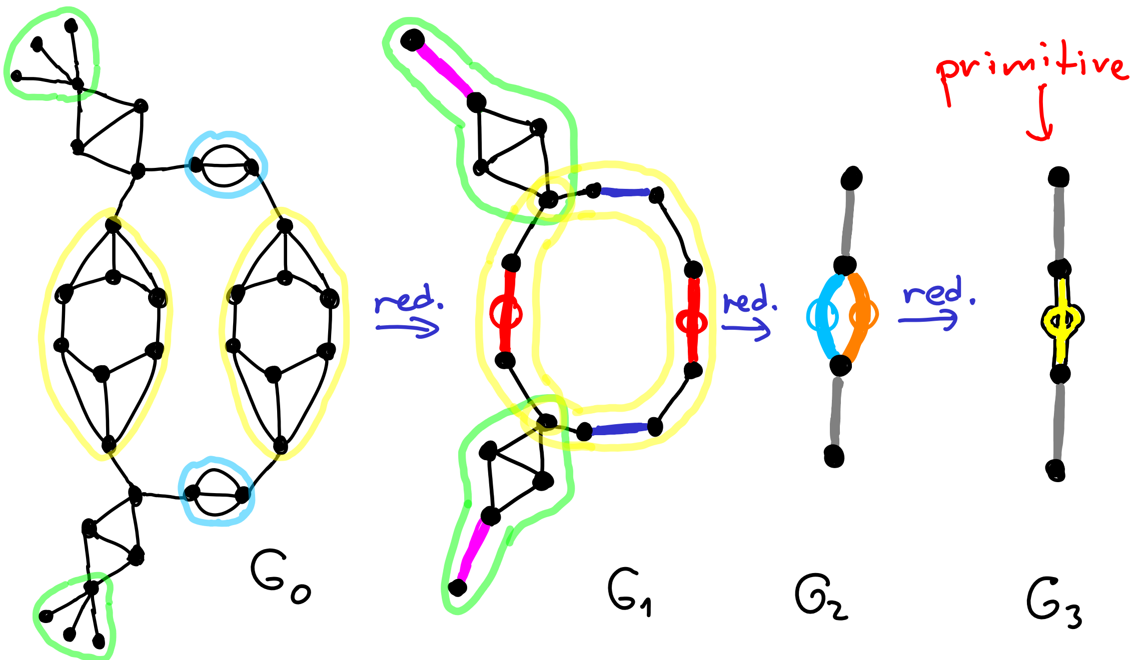
(x) Proper atoms are inclusion minimal components cutted by a 2-cut.



(x) Dipoles.



The reduction replaces atoms by colored edges which code isomorphism classes.

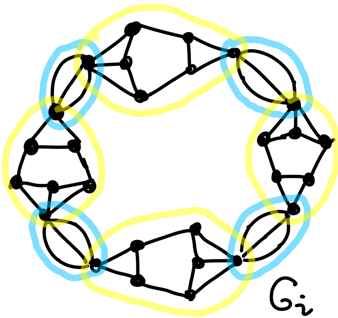


These reductions behave nicely to $\mathrm{Aut}(G_i)$.

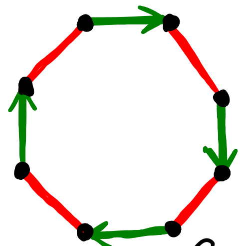
$$\boxed{\mathrm{Aut}(G_{i+1}) = \mathrm{Aut}(G_i)/\mathrm{Fix}(G_i)}$$

where $\mathrm{Fix}(A) \trianglelefteq \mathrm{Aut}(G_i)$ fixes everything except for the interiors of the atoms.

$\mathrm{Fix}(A) = \mathbb{S}_3^3 \times \mathbb{C}_2^3$ $\qquad$ $\mathrm{Aut}(G_{i+1}) = \mathbb{C}_4$

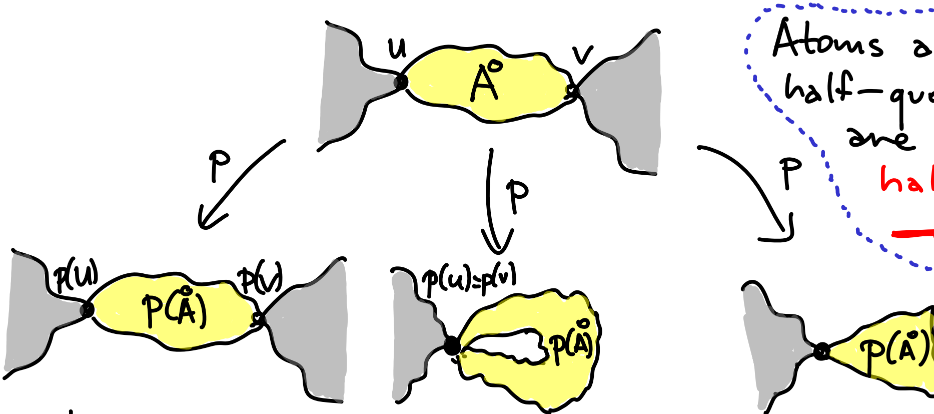


red. $\longrightarrow$



Babai 1975: Automorphism groups of planar graphs.

How do quotients of atoms look?



Atoms admitting half-quotients are called halvable.

edge-quotient

$A \cong p(A)$

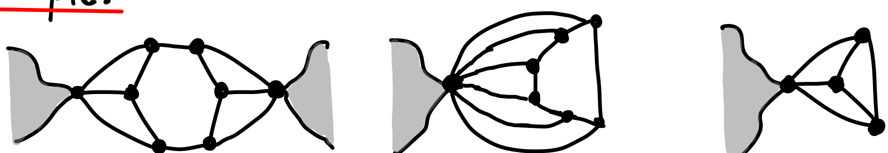
loop-quotient

$\mathring{A} \cong p(\mathring{A})$

half-quotients

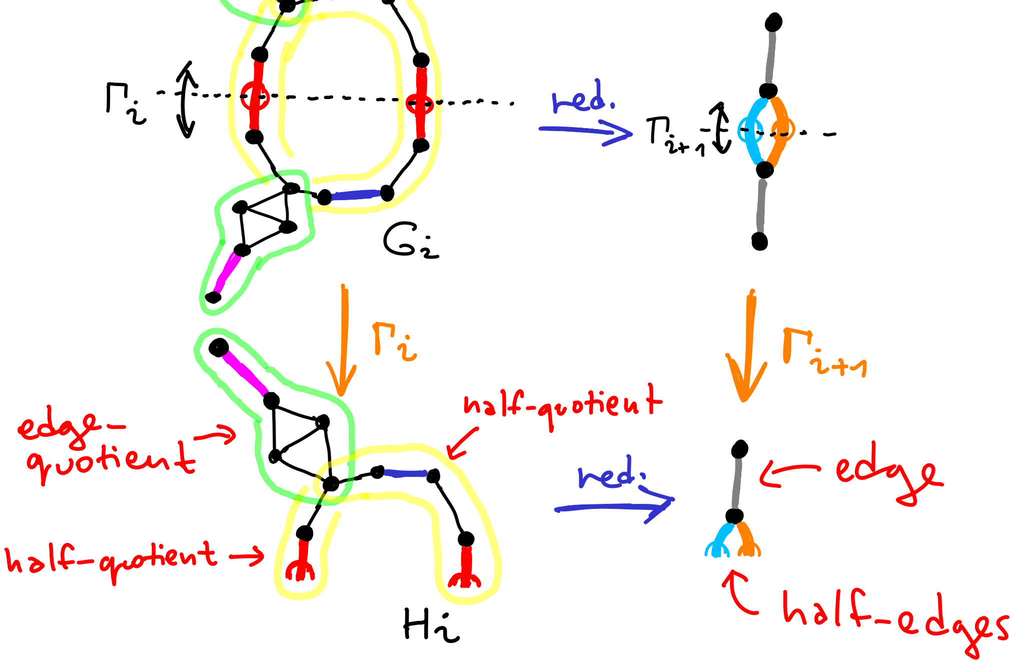
$p(A)$ is half of A.

Example.



Reductions in quotients.

$$\begin{tikzcd} G_i \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_i"] & G_{i+1} \arrow[d, "\Gamma_{i+1}"] \\ H_i \arrow[r, "\text{red.}"] & H_{i+1} \end{tikzcd}$$



Can we reverse the horizontal arrows?

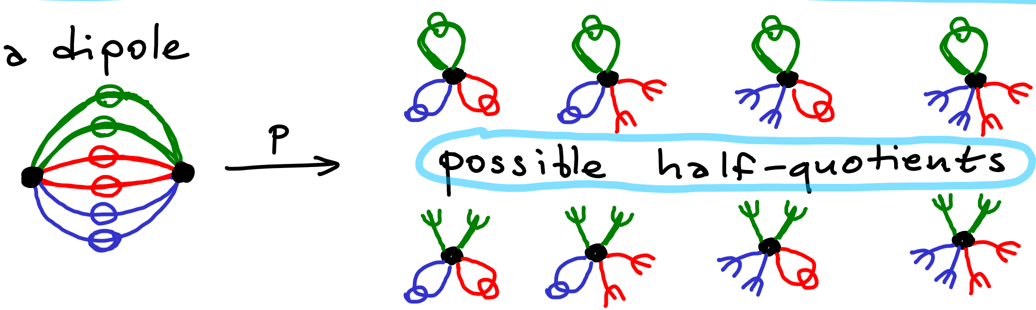
$$\begin{tikzcd} G_i \arrow[d, "\Gamma_i"] & G_{i+1} \arrow[l, "\text{exp.}"'] \arrow[d, "\Gamma_{i+1}"] \\ H_i & H_{i+1} \arrow[l, "\text{exp.}"'] \end{tikzcd}$$

?

1) Many possible Γ_i.

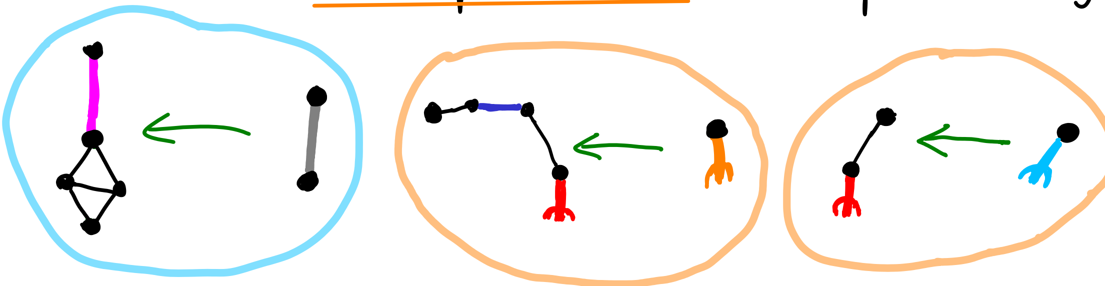
2) Many pairwise non-isomorphic H_i.

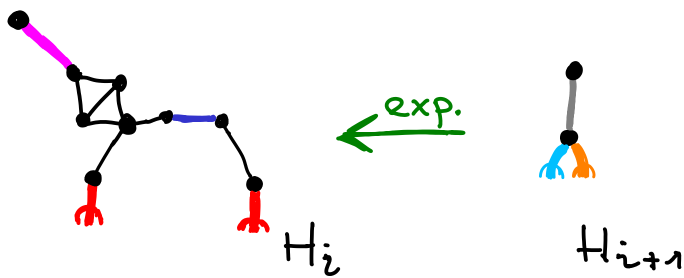
An atom can have many half-quotients.



We get the following expansion proposition:

Replace edges, loops and half-edges by the edge-quotients, the loop-quotients and some half-quotients respectively.

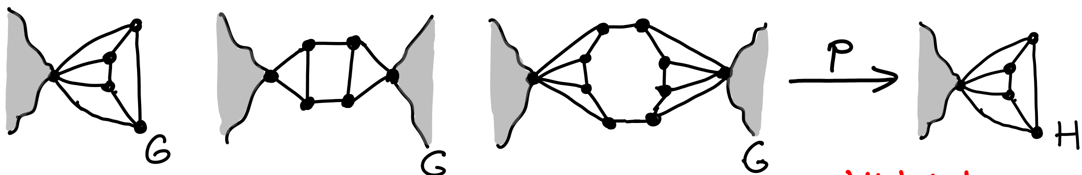




But that's not all...

$$
\begin{tikzcd}
G_0 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_0"] & G_1 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_1"] & G_2 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_2"] & \cdots \arrow[r, "\text{red.}"] & G_r \arrow[d, "\Gamma_r"] \\
H \overset{?}{\simeq} H_0 & H_1 \arrow[l, "\text{exp.}"'] & H_2 \arrow[l, "\text{exp.}"'] & \cdots \arrow[l, "\text{exp.}"'] & H_r \arrow[l, "\text{exp.}"']
\end{tikzcd}
$$

There are exponentially many possible graphs H_0.
So we need to use the structure of H.



edge-quotient loop-quotient half-quotient

Which one?
We don't know!

We solve the problem from the other side.

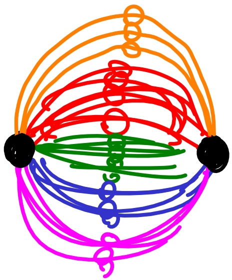
We apply reductions on H, and compute all graphs expandable to H.

The full algorithm.

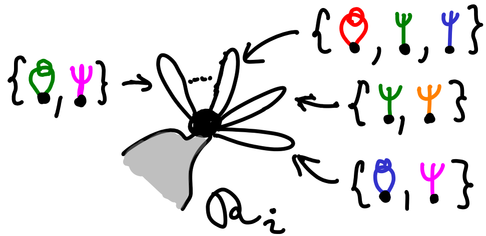
$$
\begin{tikzcd}
 & G_0 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_0"] & G_1 \arrow[r, "\text{red.}"] \arrow[d, "\Gamma_1"] & \cdots \arrow[r, "\text{red.}"] & G_r \arrow[d, "\Gamma_r"] & & \\
H \arrow[d, equal] & H_0 & H_1 \arrow[l, "\text{exp.}"'] & \cdots \arrow[l, "\text{exp.}"'] & H_r \arrow[l, "\text{exp.}"'] \arrow[r, bend left, "\text{red.}"] & \cdots \arrow[l, bend left, "\text{exp.}"] \arrow[r, bend left, "\text{red.}"] & H_s \arrow[l, bend left, "\text{exp.}"] \arrow[d, dotted, dash, "\text{match?}"] \\
\mathcal{R}_0 \arrow[r, "\text{red.}", "\text{w. lists}"'] & \mathcal{R}_1 \arrow[r, "\text{red.}", "\text{w. lists}"'] & \mathcal{R}_2 \arrow[r, "\text{red}", "\text{w. lists}"'] & \cdots \arrow[rrr, "\text{red.}", "\text{w. lists}"'] & & & \mathcal{R}_t
\end{tikzcd}
$$

One slow subroutine — half-quotients of dipoles.

a dipole



a star in $\mathcal{R}_i$ with lists



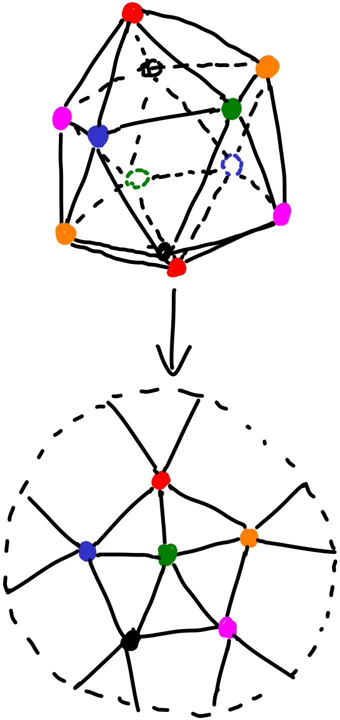
We need to decide for each color class, how many pairs form loops, and how many form half-edges.

But each loop costs two edges.

Open Problems

1. Can the slow subroutine be solved in polynomial time?
2. Is the REGULARCOVER problem equally hard as the Graph Isomorphism Problem?

Lubiw proved that testing of existence of any half-quotient is NP-hard!



Thank You!