Almost totally branched coverings between regular hypermaps

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What is a map?

**Topological map**

$\mathcal{M} = (\Gamma, S)$ is a 2-cell embedding of a connected graph $\Gamma$ into a closed surface $S$.

**Combinatorial map**

$\mathcal{M} = (D; r_0, r_1)$ where

- $D$: set of darts,
- $r_0, r_1, r_1^2 = 1$: permutations on $D$,
- $\text{Mon}(\mathcal{M}) = \langle r_0, r_1 \rangle$ is a transitive on $D$. 

Algebraic representation of maps

- Each map $\mathcal{M}$ determines a finite transitive permutation representation of Grothendieck’s cartographic group

\[ \mathcal{C}_2^+ = \langle \rho_0, \rho_1, \rho_2 \mid \rho_1^2 = \rho_0 \rho_1 \rho_2 = 1 \rangle \]

given by

\[ \theta : \mathcal{C}_2^+ \to \text{Mon}(\mathcal{M}) \leq \text{Sym}(D), \quad \rho_i \mapsto r_i, \]

where $r_2 = (r_0 r_1)^{-1}$.

- Conversely, every finite transitive permutation representation of $\mathcal{C}_2^+$ determines a map.
What is a hypermap?

**Definition (Algebraic hypermap)**

A finite transitive permutation representation of the hypercartographic group

\[ \mathcal{H}_2^+ = \langle \rho_0, \rho_1, \rho_2 \mid \rho_0\rho_1\rho_2 = 1 \rangle \cong \Delta(\infty, \infty, \infty), \]

given by

\[ \theta : \mathcal{H}_2^+ \rightarrow \text{Mon}(\mathcal{H}) \leq \text{Sym}(B), \quad \rho_i \mapsto r_i. \]

- **B**: set of brins,
- **Mon(\mathcal{H})**: monodromy group,
- the stabilizer \( H \leq \mathcal{H}_2^+ \) of a brin: hypermap subgroup,
- \((d_0, d_1, d_2)\): type, where \( d_i = o(r_i) \).
Topological hypermaps

**Definition (Topological hypermap)**
An embedding of a hypergraph into a closed (oriented) surface.

- **Hypergraph**: a set $B \neq \emptyset$ with two partitions $V$ and $E$ (hypervertices and hyperedges), and two parts are incident if they have non-empty intersection.
- **Cori representation**: $V$ and $E$ are identified with closed discs, $B = V \cap E$, and $S - (V \cup E)$ are hyperfaces.
- **Walsh map** $W(H)$: a 2-colored bipartite map corresponding to $H$. 
The Fano plane
Cori Rep. and its Walsh map $W(H)$

- Brims:
- Hyper vertices:
- Hyper edges:
- Hyper faces:

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Hypermaps and Belyĭ functions

Theorem (Belyĭ, 1979)

A compact Riemann surface $S$ is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers iff there is a meromorphic function $\beta : S \to \Sigma = \mathbb{C} \cup \{\infty\}$ with at most 3 critical values (these can be chosen as $\{0, 1, \infty\}$).

- The continuation around the critical values 0, 1 and $\infty$ of a Belyĭ function determines three permutations $r_0$, $r_1$ and $r_2$ on the sheets. These generate a transitive permutation group satisfying $r_0 r_1 r_2 = 1$, and hence, we obtain a hypermap.
- Given a hypermap on a close oriented Riemann surface $S$, one can construct a Belyĭ function on $S$ (see Jones, Singerman, 1996).
Grothendieck’s theory of dessins d’enfants

- The Absolute Galois group $\text{Gal}({\bar{\mathbb{Q}}}/\mathbb{Q})$ has a natural action on the Riemann surfaces defining Belyi functions.
- It induces a (faithful) action on the category of hypermaps and their coverings.

**Dessins d’Enfants (Children’s Drawing)**

A combinatorial approach to the study of the absolute Galois group. In other words, one can do Galois theory by drawing pictures.
Regular hypermaps

Observe that $\text{Aut}(\mathcal{H}) = C_{\text{Sym}(B)}(\text{Mon}(\mathcal{H}))$ and $\text{Mon}(\mathcal{H})$ is transitive on $B$, we have $\text{Aut}(\mathcal{H})$ is semiregular on $B$.

Definition (Regular hypermap)

$\text{Aut}(\mathcal{H})$ is regular on $B$.

- $\mathcal{H}$ is regular
  - iff $\text{Mon}(\mathcal{H})$ is regular,
  - iff $H \trianglelefteq \mathcal{H}_2^+$,

in which case

$$\text{Mon}(\mathcal{H}) \cong \mathcal{H}_2^+ / H \cong \text{Aut}(\mathcal{H}).$$
Regular coverings

Identify a regular map $\mathcal{H}$ with a quadruple $(G, x, y, z)$

- $G = \text{Aut}(\mathcal{H})$,
- $x \in G$ stabilizes a black vertex (in Walsh’s rep),
- $y \in G$ stabilizes an incident white vertex, and $z = (xy)^{-1}$,
- $\text{Mon}(\mathcal{H})$ and $\text{Aut}(\mathcal{H})$ are identified with the left and right regular representation of $G$.

Coverings between regular hypermaps

Each covering $p: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ between regular hypermaps is a regular covering, and $x_1 \mapsto x_2, y_1 \mapsto y_2, z_1 \mapsto z_2$ extends to an epimorphism $p: G_1 \rightarrow G_2$ such that $G_1/\text{CT}(p) \cong G_2$. 
Motivation

**Problem**

*Construct and classify regular hypermaps which cover the platonic maps with *cyclic* covering transformation groups,*

where *platonic maps* are defined to be regular maps \((G, x, y, z)\) of type \((d_0, 2, d_2)\),

\[
G = \langle x, y \mid x^{d_0} = y^2 = z^{d_2} = xyz = 1 \rangle,
\]

where \(1/d_0 + 1/d_2 > 1/2\).
Platonic maps

- Tetrahedron: 4 vertices, 6 edges, 4 faces.
- Cube: 8 vertices, 12 edges, 6 faces.
- Octahedron: 6 vertices, 12 edges, 8 faces.
- Icosahedron: 12 vertices, 30 edges, 20 faces.
- Dodecahedron: 20 vertices, 30 edges, 12 faces.
History

Before 21st Century certain extensions of the Platonic groups were studied in the context of polyhedral groups, by Miller (1907), Threlfall (1932), Shephard (1952), Coxeter (1940-1962), Sherk (1959).

Jones and Surowski (2000) classification of cyclic regular coverings of the Platonic maps, branched exclusively over the vertices, edges, or face-centres.


Convention

Let $\mathcal{H}_1 = (G_1, x_1, y_1, z_1) \xrightarrow{p} \mathcal{H}_2 = (G_2, x_2, y_2, z_2)$ be a covering between two regular hypermaps, where $\mathcal{H}_2$ has type $(d_0, d_1, d_2)$. Denote

$$A = \langle x_1^{d_0} \rangle, B = \langle y_1^{d_1} \rangle, C = \langle z_1^{d_2} \rangle$$

Observation

$A, B, C \leq CT(p)$ and hence $ABC \subseteq CT(p)$. 
Totally branched covering

The covering \( p : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) is called \textit{minimal} if \( CT(p) = ABC \).
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Definition

A minimal covering between two regular hypermaps is

- totally branched at hypervertices if $CT(p) = A$;

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- totally branched at hypervertices if $CT(p) = A$;
- totally branched at hyperedges if $CT(p) = B$;
- totally branched at hyperfaces if $CT(p) = C$;
- totally branched if it is (simultaneously) totally branched at hypervertices, hyperedges and hyperfaces, that is, $CT(p) = A = B = C$. 
Totally branched covering

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Almost totally branched covering

**Definition**

A minimal covering $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ between two regular hypermaps is

- almost totally branched at hypervertices if $A \triangleleft G_1$;

- almost totally branched at hyperedges if $B \triangleleft G_1$;

- almost totally branched at hyperfaces if $C \triangleleft G_1$;

- almost totally branched if it is (simultaneously) almost totally branched at hypervertices, hyperedges and hyperfaces, that is, $A, B, C \triangleleft G_1$. 
Almost totally branched covering

**Definition**

A minimal covering $\mathcal{H}_1 \rightarrow \mathcal{H}_2$ between two regular hypermaps is

- **almost totally branched at hypervertices** if $A \trianglelefteq G_1$;
- **almost totally branched at hyperedges** if $B \trianglelefteq G_1$;

where $A$ and $B$ are the groups of the hypermaps $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
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- almost totally branched at hyperfaces if $C \trianglelefteq G_1$.
Almost totally branched covering

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Observations

- Each cyclic regular hypermap covering of the platonic map is an almost totally branched covering.
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- The covering transformation group of an almost totally branched covering is a product of three cyclic groups.
Almost totally branched hypermap coverings

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- Each cyclic regular hypermap covering of the platonic map is an almost totally branched covering.
- The covering transformation group of an almost totally branched covering is a product of three cyclic groups.
- If an almost totally branched covering is smooth at one of the objects (hypervertices/hyperedges/hyperfaces), then $CT(p)$ is a metacyclic group, which can be abelian or nonabelian.
Classification in the category of maps: abelian case

**Theorem (H., Nedela, Wang, 2013)**

*Up to isomorphism, the abelian almost totally branched map-covers over the platonic maps (i.e smooth at edges), are in 1-1 correspondence with 6-tuples \((q_F, q_V, h_0, e_F, e_V, e)\) satisfying*

\[
e_V \equiv 1 \pmod{h_0}, \quad e_{F} \equiv 1 \pmod{h_0}, \\
e_V^{\text{hcf}(d_F,2)} \equiv 1 \pmod{h_0 q_V}, \quad e_F^{\text{hcf}(d_V,2)} \equiv 1 \pmod{h_0 q_F}, \\
\sum_{i=0}^{l_V-1} e_V^i \equiv 0 \pmod{q_V}, \quad \sum_{i=0}^{l_F-1} e_F^i \equiv 0 \pmod{q_F}, \\
e \sum_{i=0}^{l_F-1} e_F^i / q_F + \sum_{i=0}^{l_V-1} e_V^i / q_V \equiv 0 \pmod{h_0}, \quad e \in \mathbb{Z}_{h_0}^*.
\]

*In particular, the cyclic regular map-coverings are such maps with an extra condition \(\gcd(q_F, q_V) = 1\).*
Classification in the category of maps: nonabelian case

Theorem (H., Jones, Nedela, Wang, 2013)

- In the family of platonic maps, only the tetrahedral map, the icosahedral map and the dodecahedral map admit a nonabelian almost totally branched map-covering;
- The isomorphism classes of nonabelian almost totally branched map-coverings over an admissible platonic map are in 1-1 correspondence with the 4-tuples \((q_F, q_V, h_0, e)\) of positive integers such that
  - \(q_F, q_V\) and \(h_0\) are all even,
  - \(q_F|l_F\) and \(q_V|l_V\),
  - \(l_Fe/q_F + l_V/q_V \equiv 0 \pmod{h_0}\) where \(e \in \mathbb{Z}_{h_0}^*\).

where \(l_F\) and \(l_V\) are the numbers of faces and vertices of the platonic map.
Partial classification in the category of hypermaps

Theorem

The isomorphism classes of abelian almost totally branched hypermap-coverings over the platonic maps, smooth at faces, are in 1-1 correspondence with the solutions \((q_0, q_1, h, e_0, e_1, e)\) of the system of congruences

\[
\begin{align*}
e_0 &\equiv 1 \pmod{h}, \\
e_1 &\equiv 1 \pmod{h}, \\
e_0^{\gcd(2, d_2)} &\equiv 1 \pmod{q_0 h}, \\
e_1^{\gcd(d_0, d_2)} &\equiv 1 \pmod{q_1 h}, \\
\sum_{i=0}^{l_0-1} e_0^i &\equiv 0 \pmod{q_0}, \\
\sum_{i=0}^{l_1-1} e_1^i &\equiv 0 \pmod{q_1}, \\
e \sum_{i=0}^{l_0-1} e_0^i / q_0 + \sum_{i=0}^{l_1-1} e_1^i / q_1 &\equiv 0 \pmod{h}, \quad e \in \mathbb{Z}_h^*.
\end{align*}
\]

In particular, such a covering is cyclic iff \(\gcd(q_0, q_1) = 1\).
Open problems

1. Complete the classification of cyclic regular hypermap coverings of the platonic maps.

2. Investigation of almost totally branched coverings.
The end

Thank you very much!