Parameterized Complexity of Directed Steiner Tree and Domination Problems on Sparse Graphs

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**Steiner Tree**

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**Input:** $G = (V, E)$, $T \subseteq V$, $k \in \mathbb{N}$

**Question:** Is there a set $S \subseteq V \setminus T$, $|S| \leq k$ such that $G[S \cup T]$ is connected?

Vertices in $T$ ... terminals ... green
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On general graphs
- FPT wrt $|T|$ $(O^*(3^{|T|}))$ [Dreyfuss & Wagner 1972]
- $O^*(2^{|T|})$ time, poly-space algorithm [Nederlof 2009]
- No poly kernel wrt $|T|$ unless NP $\subseteq$ coNP/poly
- W[2]-hard wrt $k$ - easy reduction from Set Cover
- FPT wrt treewidth
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\(^1\)Fixed Parameter Tractable - there is an \( O(f(|T|) \cdot n^c) \) time algorithm

\(^2\)\( O^*(\) notation supresses polynomial factors
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\(^4\)Treewidth of a graph is the minimum width of a tree decomposition of the
Steiner Tree on Planar graphs

On planar graphs
- contract edges between terminals
- on a path at least every second vertex is Steiner
- diameter $\leq 2k$
- treewidth $O(k)$
- FPT wrt $k$
Steiner Tree on Planar graphs

On planar graphs

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FPT wrt $k$

- kernel of size $O((k + |T|)^{142})$ [Pilipczuk et al. 2013]
Intermezzo on sparse graphs

Sparse graph classes often studied:

- planar graphs
- $K_h$-minor free
- $K_h$-topological minor free
- $d$-degenerate

Sparse directed $=$ sparse underlying undirected
Directed Steiner Tree

**Input:** $D = (V, A)$, root $r \in V$, $T \subseteq V$, $k$

**Question:** Is there a set $S \subseteq V \setminus T$, $|S| \leq k$ such that in $D[S \cup T \cup \{r\}]$ there is a path from $r$ to every $t \in T$?
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On planar graphs

- cannot contract the arcs between terminals
- need a different approach
Directed Steiner Tree on planar graphs

- Contract strongly connected components
- $D[T]$ becomes a DAG
- Enough to reach the sources of this DAG
- Source-terminals, sources at least we have to find an in-neighbor for each source-terminal
- Dominators $u$ dominates $v$ iff $(u, v) \in A$
- At most $k$ dominators in solution
- Switch to $d$-degenerate big dominator $d$-dominates at least $d + 1$ sources
- Small dominator $d$-dominates at most $d$ sources
- There are "only few" big dominators but many small dominators
- Solution: ignore the small dominators!
Directed Steiner Tree on planar graphs

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\[\text{at least we have to find an in-neighbor for each source-terminal}\]
\[\text{u dominates v iff } (u, v) \in A \text{ at most } k \text{ dominators in solution}\]

Switch to $d$-degenerate big dominator
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small dominator
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Algorithm for $d$-degenerate

- look on the bipartite graph between big dominators and sources dominated by them
- $D$ is $d$-degenerate $\Rightarrow$
  - there is a source $v$ dominated by $\leq d$ dominators

Solution options:
- $v$ is dominated by some big dominator in $N^-(v)$
  - branch on which of them
- $v$ is dominated by some small dominator
  - save it for later
  - delete the big dominators in $N^-(v)$

Progress measure:
- at most $dk$ sources can be dominated by $k$ small dominators
Algorithm formally

the algorithm keeps 5 disjoint vertex sets:

- \( Y \) - partial solution - size \( \leq k \).
- \( B_h \) - dominators which dominate \( > d \) sources not dominated by \( Y \).
- \( B_l \) - dominators which dominate \( \leq d \) of them.
- \( W_h \) - sources not dominated by \( Y \), but dominated by \( B_h \).
- \( W_l \) - sources not dominated by \( Y \) or \( B_h \).

the algorithm:

- Find vertex \( v \in W_h \) with the least neighbors in \( B_h \).
- For each \( u \in B_h \cap N^-(v) \) add \( u \) to \( Y \), update the other sets, and recurse
- Put \( v \) in \( W_l \), delete \( B_h \cap N^-(v) \), update, and recurse
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the algorithm:
- Find vertex $v \in W_h$ with the least neighbors in $B_h$.
- For each $u \in B_h \cap N^-(v)$ add $u$ to $Y$, update the other sets, and recurse
- Put $v$ in $W_l$, delete $B_h \cap N^-(v)$, update, and recurse
- the measure $d(k - |Y|) - |W_l|$ drops in each case
- if $W_h$ empty, apply Nederlof’s algo with $W_l \cup Y$ as terminals
Consequences of the algorithm

- DST is FPT wrt $k$ on $d$-degenerate digraphs
- Running time $O^*(3^{kd} + o(kd))$
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- DST is FPT wrt \( k \) on \( d \)-degenerate digraphs
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But:

**Theorem**

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But:

**Theorem**


Reduction from **Set Cover**.

**Set Cover**

**Input:** A universe $U$, a family $\mathcal{F} \subseteq 2^U$, $k \in \mathbb{N}$

**Question:** Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$, such that $|\mathcal{F}'| \leq k$ and $\bigcup \mathcal{F}' = U$?
Back to $K_h$-minor-free

$K_h$-minor-free graphs

- are $O(h^2)$-degenerate
- you can do contractions
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- if each dominator dominates at least $h$ sources, then some source dominated by at most $O(h^4)$ dominators
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$K_h$-minor-free graphs

- are $O(h^2)$-degenerate
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- if each dominator dominates at least $h$ sources, then some source dominated by at most $O(h^4)$ dominators
- above bounds actually for $K_h$-topological minor free
Our results for DST

\[ D[T] \text{ arbitrary} \]

- \( O^*(3^{hk} + o(hk)) \)-time on \( K_h \)-minor free digraphs
- \( O^*(f(h)^k) \)-time on \( K_h \)-topological minor free digraphs.
- DST is \( W[2] \)-hard on 2-degenerate digraphs
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\( D[T] \) acyclic

- \( O^*(3^{hk} + o(hk)) \)-time on \( K_h \)-topological minor free digraphs
- \( O^*(3^{dk} + o(dk)) \)-time on \( d \)-degenerate graphs
  - DST is FPT wrt \( k \) on \( o(\log n) \)-degenerate graph classes
  - first FPT algorithm for undirected Steiner Tree on \( d \)-degenerate...
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  - DST is FPT wrt $k$ on $o(\log n)$-degenerate graph classes
  - first FPT algorithm for undirected $STEINER\ TREE$ on $d$-degenerate

For any constant $c > 0$, no $f(k)n^{o(\frac{k}{\log k})}$-time algorithm on graphs of degeneracy $c \log n$ unless ETH\(^5\) fails.
  - no $O^*(2^{o(d)f(k)})$-time algorithm unless ETH fails
  - no $O^*(2^{f(d)o(k)})$-time algorithm unless ETH fails

\(^5\)Exponential time hypothesis — 3-SAT cannot be solved in time $2^{o(n)}$
Application to Dominating Set

- there is a parameterized reduction from Dominating Set to (Directed) Steiner Tree, preserving the degeneracy
- the algorithm can be adapted - using the above reduction when there only a few vertices left to be dominated
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The sets

- $Y$ - partial solution - size $\leq k$.
- $B$ - vertices dominated by $Y$.
- $W$ - not dominated by $Y$.
- $B_h$ - vertices of $B$ dominating $\geq d + 1$ vertices of $W$.
- $B_l$ - vertices of $B$ dominating $\leq d$ vertices of $W$.
- $W_h$ - vertices in $W$ with neighbor in $B_h \cup W$.
- $W_l$ - remaining vertices of $W$. 
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[Alon & Gutner 2009]
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- No algorithm $O^*(2^{o(d)f(k)})$, unless ETH fails.
- No algorithm $O^*(2^{f(d)o(k)})$, unless ETH fails.
- $O(dn \log n)$-time $d^2$-approximation algorithm on $d$-degenerate graphs.

[Alon & Gutner 2009]
Future research, Open questions

- Asymptot. optimal running times for $d$-degenerate graphs - $2^{O(kd)}$
  - SETH lower bound on the basis?
  - Improving the upper bound - currently $3^{kd+o(kd)}$

- **Strongly Connected Steiner Subgraph**, **Directed Steiner Network** in planar graphs

- kernel for planar Steiner Tree wrt $|T|$? wrt $k$?
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Asymptot. optimal running times for $d$-degenerate graphs - $2^{O(kd)}$

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- Strongly Connected Steiner Subgraph,
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- Kernel for planar Steiner Tree wrt $|T|$? wrt $k$?

$^6$Strong ETH — SAT cannot be solved in time $(2 - \varepsilon)^n$
Thank you for your attention!