The probability of planarity of a random graph near the critical point

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The material of this talk

1. — Planarity on the critical window for random graphs
2. — Our result. The strategy
3. — Cubic planar multigraphs
4. — Other applications
Planarity on the critical window for random graphs
The model $G(n, M)$

There are $2^{\binom{n}{2}}$ labelled graphs with $n$ vertices.

A random graph $G(n, M)$ is the probability space with properties:

- **Sample space**: set of labelled graphs with $n$ vertices and $M = M(n)$ edges.
- **Probability**: Uniform probability $(\binom{n}{2}^{-1})$

Properties:

- ♢ Fixed number of edges ✓
- ♣ The probability that a fixed edge belongs to the random graph is $p = \binom{n}{2}^{-1} M$. ✓
- ♠ There is not independence.
The Erdős-Rényi phase transition

Random graphs in $G(n, M)$ present a dichotomy for $M = \frac{n}{2}$:

1. **(Subcritical)** $M = cn$, $c < \frac{1}{2}$: a.a.s. all connected components have size $O(\log n)$, and are either trees or unicyclic graphs.

2. **(Critical)** $M = \frac{n}{2} + \lambda n^{2/3}$: a.a.s. the largest connected component has size of order $n^{2/3}$

3. **(Supercritical)** $M = cn$, $c > \frac{1}{2}$: a.a.s. there is a unique component of size of order $n$.

Double jump in the creation of the *giant component*. 
**The problem; what was known**

**ON THE EVOLUTION OF RANDOM GRAPHS**

by

P. ERDŐS and A. RÉNYI

Dedicated to Professor P. Turán at his 50th birthday.

We can show that for $N(n) = \frac{n}{2} + \lambda \sqrt{n}$ with any real $\lambda$ the probability of $\Gamma_{n,N(n)}$ not being planar has a positive lower limit, but we cannot calculate its value. It may even be 1, though this seems unlikely.

**PROBLEM: Compute**

$$p(\lambda) = \lim_{n \to \infty} \Pr \{ G \left( n, \frac{n}{2} (1 + \lambda n^{-1/3}) \right) \text{ is planar} \}$$

What was known:

- Janson, Łuczak, Knuth, Pittel (94): $0.9870 < p(0) < 0.9997$
- Łuczak, Pittel, Wierman (93): $0 < p(\lambda) < 1$

Our contribution: the whole description of $p(\lambda)$
Our result. The strategy.
The main theorem

**Theorem (Noy, Ravelomanana, R.)** Let $g_r$ be the number of cubic planar weighted multigraphs with $2r$ vertices. Write

$$A(y, \lambda) = \frac{e^{-\lambda^3/6}}{3(y+1)/3} \sum_{k \geq 0} \frac{(\frac{1}{2}3^{2/3}\lambda)^k}{k! \Gamma((y + 1 - 2k)/3)}.$$

Then the limiting probability that the random graph $G\left(n, \frac{n}{2}(1 + \lambda n^{-1/3})\right)$ is planar is

$$p(\lambda) = \sum_{r \geq 0} \frac{\sqrt{2\pi}}{(2r)!} g_r A\left(3r + \frac{1}{2}, \lambda\right).$$

In particular, the limiting probability that $G\left(n, \frac{n}{2}\right)$ is planar is

$$p(0) = \sum_{r \geq 0} \sqrt{\frac{2}{3}} \left(\frac{4}{3}\right)^r g_r \frac{r!}{(2r)!^2} \approx 0.99780.$$
The strategy (I): pruning a graph

The resulting multigraph is the core of the initial graph.
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The strategy (and II): shape in the critical window

Łuczak, Pittel, Wierman (94):
the structure of a random graph in the critical window

\[
p(\lambda) = \frac{\text{number of planar graphs with } \frac{n}{2}(1 + \lambda n^{-1/3}) \text{ edges}}{\binom{n}{2}(1 + \lambda n^{-1/3})}
\]

Hence... \textbf{We need to count!}
The symbolic method à la Flajolet

COMBINATORIAL RELATIONS between CLASSES

↕⇕↕

EQUATIONS between GENERATING FUNCTIONS

<table>
<thead>
<tr>
<th>Class</th>
<th>Relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C = A \cup B$</td>
<td>$C(x) = A(x) + B(x)$</td>
</tr>
<tr>
<td>$C = A \times B$</td>
<td>$C(x) = A(x) \cdot B(x)$</td>
</tr>
<tr>
<td>$C = \text{Seq}(B)$</td>
<td>$C(x) = (1 - B(x))^{-1}$</td>
</tr>
<tr>
<td>$C = \text{Set}(B)$</td>
<td>$C(x) = \exp(B(x))$</td>
</tr>
<tr>
<td>$C = A \circ B$</td>
<td>$C(x) = A(B(x))$</td>
</tr>
</tbody>
</table>

All GF are exponential ≡ labelled objects

$$A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$
Trees

We apply the previous grammar to count *rooted* trees

\[ T = \bullet \times \text{Set}(\mathcal{T}) \to T(x) = xe^{T(x)} \]

To forget the root, we just integrate: \((xU'(x) = T(x))\)

\[
\int_0^x \frac{T(s)}{s} ds = \left\{ \begin{array}{l}
T(s) = u \\
T'(s) \, ds = du
\end{array} \right\} = \int_{T(0)}^{T(x)} 1 - u \, du = T(x) - \frac{1}{2} T(x)^2
\]

and the general version

\[ e^{U(x)} = e^{T(x)} e^{-\frac{1}{2}T(x)^2} \]
Unicyclic graphs

\[ V = \bigcirc \geq 3 (T) \rightarrow V(x) = \sum_{n=3}^{\infty} \frac{1}{2} \frac{(n-1)!}{n!} (T(x))^n \]

We can write \( V(x) \) in a compact way:

\[ \frac{1}{2} \left( - \log (1 - T(x)) - T(x) - \frac{T(x)^2}{2} \right) \rightarrow e^{V(x)} = \frac{e^{-T(x)/2 - T(x)^2/4}}{\sqrt{1 - T(x)}}. \]
Cubic planar multigraphs
Planar graphs arising from cubic multigraphs

In an informal way:

\[ G(\bullet \leftarrow T, \bullet - \bullet \leftarrow \text{Seq}(T)) \]
Weighted planar cubic multigraphs

Cubic multigraphs have $2r$ vertices and $3r$ edges (Euler’s Relation)

$$G(x, y) = \sum_{r \geq 1} \frac{g_r}{(2r)!} x^{2r} y^{3r} = g(x^2 y^3)$$

We need to remember the number of loops and the number of multiple edges to avoid symmetries:

weights $2^{-f_1-f_2} (3!)^{-f_3}$
The equations

We have equations defining $G(z)$:

\[
\begin{align*}
G(z) &= \exp G_1(z) \\
3z \frac{dG_1(z)}{dz} &= D(z) + C(z) \\
B(z) &= \frac{z^2}{2} (D(z) + C(z)) + \frac{z^2}{2} \\
C(z) &= S(z) + P(z) + H(z) + B(z) \\
D(z) &= \frac{B(z)^2}{z^2} \\
S(z) &= C(z)^2 - C(z)S(z) \\
P(z) &= z^2C(z) + \frac{1}{2}z^2C(z)^2 + \frac{z^2}{2} \\
2(1 + C(z))H(z) &= u(z)(1 - 2u(z)) - u(z)(1 - u(z))^3 \\
z^2(C(z) + 1)^3 &= u(z)(1 - u(z))^3.
\end{align*}
\]

$u(z)$ is the INPUT: arising from map enumeration.
The equations: an appetizer

All GF obtained (except $G(z)$) are algebraic GF; for instance:

\[
1048576 z^6 + 1034496 z^4 - 55296 z^2 + \\
(9437184 z^6 + 6731264 z^4 - 1677312 z^2 + 55296) C + \\
(37748736 z^6 + 18925312 z^4 - 7913472 z^2 + 470016) C^2 + \\
(88080384 z^6 + 30127104 z^4 - 16687104 z^2 + 1622016) C^3 + \\
(132120576 z^6 + 29935360 z^4 - 19138560 z^2 + 2928640) C^4 + \\
(132120576 z^6 + 19314176 z^4 - 12429312 z^2 + 2981888) C^5 + \\
(88080384 z^6 + 8112384 z^4 - 4300800 z^2 + 1720320) C^6 + \\
(37748736 z^6 + 2097152 z^4 - 614400 z^2 + 524288) C^7 + \\
(9437184 z^6 + 262144 z^4 + 65536) C^8 + 1048576 C^9 z^6 = 0.
\]
The estimates

- The **excess** of a graph \( ex(G) \) is the number of edges minus the number of vertices.

\[
\begin{align*}
n! [z^n] & \quad \text{Trees, } ex = -1 \quad \text{Unicyclic, } ex = 0 \quad \text{Cubic, } ex = 3r - 2r = r \\
& \quad \frac{U(z)^{n-M+r}}{(n-M+r)!} \quad \frac{e^{-T(z)/2-T(z)^2/4}}{\sqrt{1 - T(z)}} \quad \frac{g_r T(z)^{2r}}{(1 - T(z))^{3r}}
\end{align*}
\]

- We finally use **saddle point estimates**
Other applications
General families of graphs

Many families of graphs admit a straightforward analysis:

(Noy, Ravelomanana, R.)

Let $G = \text{Ex}(H_1, \ldots, H_k)$ and assume all the $H_i$ are 3-connected. Let $h_r$ be the number of cubic multigraphs in $G$ with $2r$ vertices. Then the limiting probability that the random graph $G(n, \frac{n}{2}(1 + \lambda n^{-1/3}))$ is in $G$ is

$$p_G(\lambda) = \sum_{r \geq 0} \frac{\sqrt{2\pi}}{(2r)!} h_r A \left( 3r + \frac{1}{2}, \lambda \right).$$

In particular, the limiting probability that $G(n, \frac{n}{2})$ is in $G$ is

$$p_G(0) = \sum_{r \geq 0} \sqrt{\frac{2}{3}} \left( \frac{4}{3} \right)^r h_r \frac{r!}{(2r)!^2}.$$

Moreover, for each $\lambda$ we have

$$0 < p_G(\lambda) < 1.$$
Examples...please

Some interesting families fit in the previous scheme:

- **Ex**\((K_4)\): series-parallel graphs: there are not 3-connected elements in the family!

- **Ex**\((K_{3,3})\): The same limiting probability as planar... \(K_5\) does not appear as a core!

- Many others: **Ex**\((K_{3,3}^+)\), **Ex**\((K_5^-)\), **Ex**\((K_2 \times K_3)\)...

- **PROBLEM**: coloured graphs (in preparation...)

- **PROBLEM**: compute exactly for graphs on surfaces
Gràcies!
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