Katětov expanders

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Motivation

**Definition (Katětov 1986)**

Let \((X, \varrho)\) be a metric space. A **Katětov function** on \(X\) is a map \(f : X \to \mathbb{R}\) satisfying

1. \(|f(x) - f(y)| \leq \varrho(x, y)\)
2. \(\varrho(x, y) \leq f(x) + f(y)\)

for every \(x, y \in X\).

Denote by \(K(X)\) the space of all Katětov functions on \(X\), endowed with the sup metric.

**Claim**

The construction above extends to a self-functor on the category of metric spaces with nonexpansive maps.
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Claim

The construction above extends to a self-functor on the category of metric spaces with nonexpansive maps.
\[ \begin{array}{ccc}
X & \xrightarrow{\eta_X} & K(X) \\
\downarrow f & & \downarrow K(f) \\
Y & \xrightarrow{\eta_Y} & K(Y)
\end{array} \]
General assumptions:

We fix a class $\mathcal{I}$ of *small* models of a fixed type. We assume:

- $\mathcal{I}$ has the joint embedding property
- $\mathcal{I}$ has the amalgamation property
- $\mathcal{I}$ is closed under isomorphisms

Notation:

$\sigma \mathcal{I}$ will denote the class of all structures isomorphic to unions of countable chains in $\mathcal{I}$. 
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Theorem (Fraïssé 1954)

Assume $\mathcal{S}$ is countable. Then there exists a unique $\mathcal{S}$-homogeneous structure $U \in \sigma \mathcal{S}$.

$U$ is often called the Fraïssé limit of $\mathcal{S}$.

$\mathcal{S}$ is called a Fraïssé class.
Problem ((?) Jaligot, 2007)

Let $\mathcal{I}$ be a Fraïssé class with the Fraïssé limit $U$. Is it always true that $\text{Aut}(U)$ is universal for the class $\{\text{Aut}(X) : X \in \sigma\mathcal{I}\}$?
Some references


Let $\mathcal{I} \subseteq \sigma \mathcal{I}$ be as before, now treated as categories.

**Definition**

An expander on $\langle \mathcal{I}, \sigma \mathcal{I} \rangle$ is a pair $\langle F, \eta \rangle$, where $F: \mathcal{I} \to \sigma \mathcal{I}$ is a covariant functor and $\eta$ is a natural transformation from $\text{id}_\mathcal{I}$ to $F$. 

\[ 
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & F(A) \\
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An **expander** on \( \langle \mathcal{I}, \sigma \mathcal{I} \rangle \) is a pair \( \langle F, \eta \rangle \), where \( F : \mathcal{I} \to \sigma \mathcal{I} \) is a covariant functor and \( \eta \) is a natural transformation from id\( \mathcal{J} \) to \( F \).

![Diagram](http://www.math.cas.cz/kubis/)

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Katětov expanders

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Lemma

Every expander on $\langle \mathcal{I}, \sigma \mathcal{I} \rangle$ extends to a continuous expander on $\langle \sigma \mathcal{I}, \sigma \mathcal{I} \rangle$.

Continuity means that

$$ F(\lim_{n \to \infty} X_n) = \lim_{n \to \infty} F(X_n) $$

whenever $X_0 \subseteq X_1 \subseteq \ldots$ is a tower of $\sigma \mathcal{I}$-structures.
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Katětov expanders

Fix a family $\mathcal{F}$ of embeddings between $\mathcal{I}$-objects so that every embedding $j: A \to B$ with $A, B \in \mathcal{I}$ is of the form

$$j = e_1 \circ \ldots \circ e_n$$

for some $e_1, \ldots, e_n \in \mathcal{F}$.

**Definition**

An expander $\langle K, \eta \rangle$ is Katětov with respect to $\mathcal{F}$ if for every $X \in \mathcal{L}$, for every embeddings $f: A \to X$, $e: A \to B$ with $A, B \in \mathcal{I}$ and $e \in \mathcal{F}$, there exists an embedding $\bar{f}: B \to K(X)$ such that $\bar{f} \circ e = \eta_X \circ f$. In other words, the following diagram is commutative:
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$$
\begin{array}{c}
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A & \ar[l]^{e} B \ar[u]_{\bar{f}}}
$$
Fact

Let $K$ be a Katětov expander on $\mathcal{I}$. Then $K^\omega(X)$ is the Fraïssé limit of $\mathcal{I}$ for every $X \in \sigma \mathcal{I}$.

Here,

- $K^\omega(X) = \lim_{n \to \infty} K^n(X)$,
- $K^n(X) = K(K^{n-1}(X))$, $K^0(X) = X$. 

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Automorphism groups

**Theorem**

Assume $K$ is a Katětov expander on $\mathcal{I}$ and let $U$ be the Fraïssé limit of $\mathcal{I}$. Then for every $X \in \sigma \mathcal{I}$ the natural embedding

$$X \hookrightarrow K(X)$$

induces an embedding $\text{Aut}(X) \hookrightarrow \text{Aut}(U)$.

If $K$ is an expander on homomorphisms, then the same holds for the endomorphism semigroups.
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Pushouts

Definition

A class of structures $\mathcal{S}$ admits pushouts if for every embeddings $i: C \to A$, $j: C \to B$ in $\mathcal{S}$, there exist embeddings $i': A \to D$, $j': B \to D$ satisfying $i' \circ i = j' \circ j$ and

- for every homomorphisms $f: A \to X$, $g: B \to X$ with $f \circ i = g \circ j$, there exists a unique homomorphism $h: D \to X$ such that $h \circ i' = f$ and $h \circ j' = g$. 

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The pushout of $\langle i, j \rangle$
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Existence result

Theorem

Assume $\mathcal{I}$ admits pushouts. Then there exists a Katětov expander in $\langle \mathcal{I}, \sigma \mathcal{I} \rangle$.

Remark

If, additionally, $\mathcal{I}$ has mixed pushouts then there exists a Katětov expander for all homomorphisms. Consequently, denoting by $U$ the Fraïssé limit of $\mathcal{I}$, the semigroup $\text{End}(U)$ is universal for $\{ \text{End}(X) : X \in \sigma \mathcal{I} \}$.
Existence result

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Corollary

The following Fraïssé classes of finite structures admit Katětov expanders:

- graphs
- directed graphs
- $K_n$-free graphs
- posets
- semilattices

Remark

The class of $K_n$-free graphs ($n > 2$) does not admit a Katětov expander for homomorphisms. This follows from a result of Mudrinski (2010): The $K_n$-free Henson graph is retract-rigid (identity is the only retraction).
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Other examples of Katětov expanders

Example

Let $\mathcal{I}$ be the class of finite linearly ordered sets. Given $S \in \mathcal{I}$, define

$$K(S) = S \cup \text{hom}(S, \{0, 1\}),$$

with the natural linear ordering.

Example

Let $\mathcal{I}$ be the class of all finite groups. Given $G \in \mathcal{I}$ let $F(G)$ be the group of all permutations of the set $G$. Identifying $G$ with the subgroup of $F(G)$, we can extend $F$ to an expander.

Claim

$F^\omega$ is a Katětov expander.
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$F^\omega$ is a Katětov expander.
Proposition

There exists a Katětov expander in the class of finite tournaments.

Proof.

Given a finite tournament $T$, let $K(T)$ be the set of one-to-one sequences in $T$, agreeing that the empty sequence dominates everything.

Identify $T$ with sequences of length 1.

Endow $K(T)$ with the lexicographic tournament structure.
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Problem

Does there exist a Fraïssé class $\mathcal{F}$ with no Katětov expander?

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When does there exist a Katětov expander for homomorphisms?
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