Cubic vertices in minimal bricks

Andrea Jiménez

Instituto de Matemática e Estatística, Universidade de São Paulo

Joint work with Maya Stein
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- $\mathcal{M}(G) := \text{all perfect matchings of } G$, $\mathcal{M}(G) \neq \emptyset$ [Tutte 1947]
  
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- Perfect matching polytope of $G$ [Edmonds 1965]

Tight cut decomposition procedure [Kotzig 59, and Lovász & Plummer 72]
- Output: A list of graphs without non-trivial tight cuts.
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A cut $C$ of $G$ is **tight** if every perfect matching of $G$ has exactly one edge in $C$. 
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Theorem [Edmonds, Lovász & Pulleyblank and Lovász 80’s ]

$G$ does not have non-trivial tight cuts if and only if $G$ is a \textit{brick} or a \textit{brace}.

Theorem [Lovász 1986]
The list of bricks and braces is unique.
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$G$ matching-covered graph, $b :=$ number of bricks and

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Pfaffian Orientations

An orientation of a graph $G$ is Pfaffian if for every perfect matching $M$ of $G$ each even cycle of $G \setminus M$ has an odd number of edges directed in either direction.
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Theorem [Kasteleyn 1967]

If $G$ is a Pfaffian graph, then $|\mathcal{M}(G)|$ can be computed in polynomial time.
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- Pfaffian bricks — Norine (Ph.D. Thesis) 2005
Theorem [Carvalho, Lucchesi & Murty 2004]

Every brick can be obtained from one of the basic bricks by a sequence of applications of the following four operations (expansions):

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Corollary (Lovász’s Conjecture)

Every minimal brick has a vertex of degree 3.
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- More about brick generation — Norine & Thomas
Every minimal brick other than the Petersen graph can be obtained from $K_4$ or $	ilde{C}_6$ by a sequence of applications of **strict extensions**.

### Theorem [Norine & Thomas 2005]

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<th>strict linear 3</th>
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Every minimal brick $G$ has at least $\frac{|V(G)|}{9}$ vertices of degree 4.
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Theorem [J. & Stein 2013]
Every minimal brick $G$ has at least $\frac{1}{52} \sqrt{|V(G)|}$ vertices of degree 3.
Proof’s Ideas

- $G_0 \xleftarrow{\psi_1} G_1 \xleftarrow{\psi_2} \ldots \xleftarrow{\psi_k} G_k := \text{min-brick sequence} \ [\text{Norine} \& \text{Thomas}]$

- $\{\psi_1, \psi_2, \ldots, \psi_k\}$ strict extensions, $G = G_k$
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- \{\psi_1, \psi_2, \ldots, \psi_k\} \text{ strict extensions, } G = G_k
- |G_i|_3 := \text{number of cubic vertices in } G_i$

Generous, neutral and selfish operations

$i \in \{1, \ldots, k\}, \quad p = p(i) = |G_i|_3 - |G_{i-1}|_3$
- **Generous** if $p > 0$, **Neutral** if $p = 0$, **Selfish** if $p < 0$
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- |G|_3 := \text{number of cubic vertices in } G
- If $G$ is such that $d(G) \leq 4 - \gamma$ and $\delta \geq 3$, then $|G|_3 \geq \gamma|V(G)|$. 

Other details:

- $d(i) = 5$
- $d(i) = 4$
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Average degree

$i \in \{1, \ldots, k\}$ we define

\[ n(i) := |V(G_i)| - |V(G_{i-1})| \quad e(i) := |E(G_i)| - |E(G_{i-1})| \quad d(i) := 2 \frac{e(i)}{n(i)} \]
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- \( G_0 \xrightarrow{\psi_1} G_1 \xrightarrow{\psi_2} \ldots \xrightarrow{\psi_k} G_k := \) nice min-brick sequence

- \( I_s \subset \{1, 2, \ldots, k\} \), with \( i \in I_s \) for \( \psi_i \) selfish
- $G_0 \xrightarrow{\psi_1} G_1 \xrightarrow{\psi_2} \ldots \xrightarrow{\psi_k} G_k := \text{nice min-brick sequence}$

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Case 1: $|I_s| \geq \frac{1}{2} \sqrt{k}$
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Lemma

There exists a partition $I_s^a, I_s^b$ of $I_s$ such that

(a) for each $i \in I_s^a$ there is a vertex $v_i$ that has degree 3 in $G$ and the $v_i'$s are distinct for distinct $i \in I_s^a$, and

(b) there is $\tilde{I}_s^b \subset \{1, \ldots, k\}$ such that $I_s^b \subseteq \tilde{I}_s^b$ and

$$\sum_{j \in \tilde{I}_s^b} (|G_j|_3 - |G_{j-1}|_3) \geq \frac{1}{4} |I_s^b|.$$
Case 2: \( |I_s| < \frac{1}{2} \sqrt{k} \)
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- \(I_n \subset \{1, 2, \ldots, k\}\), with \(j \in I_n\) for \(\psi_j\) neutral and \(d(j) = 4\)
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- $I_n \subset \{1, 2, \ldots, k\}$, with $j \in I_n$ for $\psi_j$ neutral and $d(j) = 4$

Case 2.1: $|I_n| \geq k - \frac{27}{26} \sqrt{k}$
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- (i) there exist a bad subsequence of length at least $\frac{27}{52} \sqrt{k}$
  - (ii) subcase (i) does not happen.
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Case 2: \( |I_s| < \frac{1}{2} \sqrt{k} \)

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  (ii) \(\sim\) Case 1; taking bad subsequences instead of isolated operations.
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Case 2.2: \(|\{1, \ldots, k\} - I_s - I_n| \geq \frac{7}{13}\sqrt{k}\)
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Case 2.2: \( |\{1, \ldots, k\} - I_s - I_n| \geq \frac{7}{13} \sqrt{k} \)

- \( i \in |\{1, \ldots, k\} - I_s - I_n| \), then \( d(i) \leq 3.5 \)
Gracias :-}