Fractional colorings and independent sets in cubic graphs with large girth

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joint work with František Kardoš and Daniel Král'

Definitions

Independent set I of a graph G

- $I \subseteq V(G)$ s.t. no two vertices from I are adjacent
- $\alpha(G)$ is the size of the largest independent set in G

Fractional coloring of a graph G

- an assignment of non-neg. weights to independent sets of G s.t. for each vertex sum of sets containing it is at least one
- $\chi_f(G)$ is the minimum sum of weights over all such colorings

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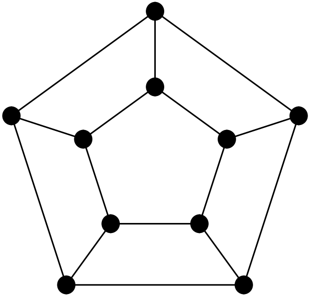
Example



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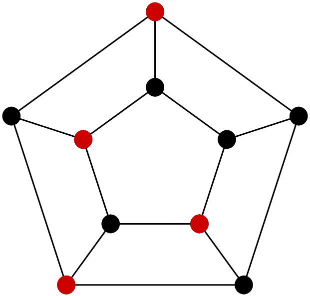

the independence number

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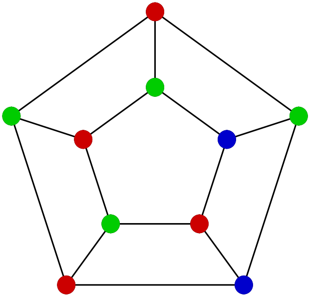

the chromatic number

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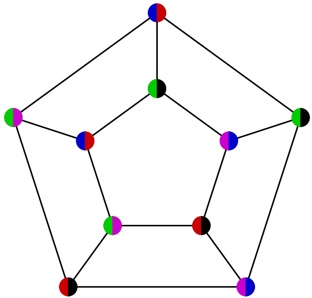

the fractional chromatic number

Previous and related results

Fractional chromatic number of cubic graph with large girth

- closely related to Nešetřil's Pentagon Conjecture ($\chi_c \leq 2.5$)
- $\chi_f(G) \leq \frac{8}{3} = 2.6667$ by Hatami and Zhu (2009)
- $\chi_f(G) \leq 2.2978$ (our result)

independent sets cubic graphs with large girth

- $\alpha(G) \geq 0.4328n - o(n)$ by Hoppen (2008)
- $\alpha(G) \geq 0.4352n$ (our result)

independent sets in random cubic graphs

- $\alpha(G) \geq 0.43475n$ by Duckworth, Zito (2009)
- $\alpha(G) \leq 0.455n$ by McKay (1987)

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Basic idea

Process graph in rounds, construct distribution on indep. sets

First round

- randomly and independently choose some vertices
- put them into indep. set, remove them and their neighbors

Other rounds

- paths with endpoints of deg. 1/3, inner vertices have deg. 2
- paths between vertices of degree 1 process optimally
- greedily process paths between endpoints of deg. 1 and 3
- choose some paths between vertices of degree 3,
  the probability depends on their length,
  inner part of choosen paths process optimally

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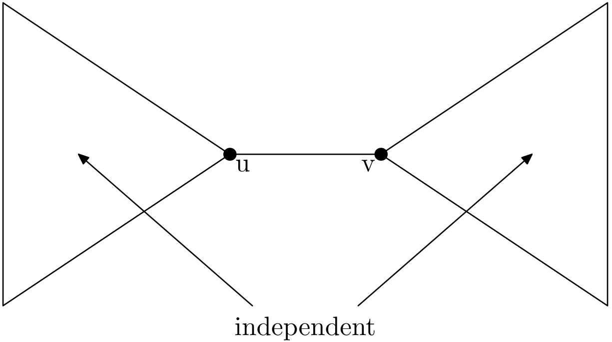
Independence lemma



Recurrence relations

$$\psi_k = \left(\widehat{P_k^{\to 3}} + P_k^{E\to 1} + \frac{P^{\widetilde{O\to 3}}{}_k}{2} \right)^2$$

$$F_k\left(3,3,2,2\to 3\right) = \psi_k \left(\frac{(1-p_2)\cdot \widehat{P_k^{\to 3}}}{P^{O\to 1} + \frac{1}{2}p_2 P_k^{E\to 3} + (1-p_2)\cdot\left(\frac{1}{2}\widetilde{P_k^{E\to 3}} + \widehat{P_k^{\to 3}}\right)} \right)^2$$

$$F_k\left(3,3,3,3\to 3\right) = \left(\psi_k\right)^3$$

$$\vdots$$

$$H_k\left(3,3,2,2\right) = w_k^3 q_k^3 \left(q_k^2\right)^2 \cdot \left(P_k^{O\to 1} + \frac{1}{2}p_2 P_k^{E\to 3} + (1-p_2)\cdot\left(\frac{1}{2}\widetilde{P_k^{E\to 3}} + \widehat{P_k^{\to 3}}\right)\right)^2$$

$$H_k\left(3,3,3,3\right) = w_k^3\left(q_k^3\right)^3$$

$$\vdots$$

$$w_{k+1}^1 = \left(\sum_{*} F_k(*\to 1)H_k(*)\right) \Big/ \left(\sum_{*,i} F_k(*\to i)H_k(*)\right)$$

$$w_{k+1}^2 = \left(\sum_{*} F_k(*\to 2)H_k(*)\right) \Big/ \left(\sum_{*,i} F_k(*\to i)H_k(*)\right)$$

$$w_{k+1}^3 = \left(\sum_{*} F_k(*\to 3)H_k(*)\right) \Big/ \left(\sum_{*,i} F_k(*\to i)H_k(*)\right)$$

Main results

Theorem

Every cubic graph with sufficiently large girth has the fractional chromatic number at most 2.2978.

Corollary

Every cubic graph with sufficiently large girth contains an independent set of size $0.4352n$.

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Random cubic graph a.a.s contains an independent set of size $0.4352n$.

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