Fractional colorings and independent sets in cubic graphs with large girth

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joint work with František Kardoš and Daniel Král’
Definitions

**Independent set** $I$ of a graph $G$

- $I \subseteq V(G)$ s.t. no two vertices from $I$ are adjacent
- $\alpha(G)$ is the size of the largest independent set in $G$

**Fractional coloring of a graph** $G$

- an assignment of non-neg. weights to independent sets of $G$ s.t. for each vertex sum of sets containing it is at least one
- $\chi_f(G)$ is the minimum sum of weights over all such colorings
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Example
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the independence number
Example

the chromatic number
Example

the fractional chromatic number
Previous and related results

Fractional chromatic number of cubic graph with large girth

- closely related to Nešetřil’s Pentagon Conjecture ($\chi_c \leq 2.5$)
  - $\chi_f(G) \leq \frac{8}{3} = 2.6667$ by Hatami and Zhu (2009)
  - $\chi_f(G) \leq 2.2978$ (our result)

Independent sets cubic graphs with large girth

- $\alpha(G) \geq 0.4328n - o(n)$ by Hoppen (2008)
- $\alpha(G) \geq 0.4352n$ (our result)

Independent sets in random cubic graphs

- $\alpha(G) \geq 0.43475n$ by Duckworth, Zito (2009)
- $\alpha(G) \leq 0.455n$ by McKay (1987)
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Basic idea

Process graph in rounds, construct distribution on indep. sets

First round

- randomly and independently choose some vertices
- put them into indep. set, remove them and their neighbors

Other rounds

- paths with endpoints of deg. 1/3, inner vertices have deg. 2
- paths between vertices of degree 1 process optimally
- greedily process paths between endpoints of deg. 1 and 3
- choose some paths between vertices of degree 3, the probability depends on their length, inner part of choosen paths process optimally
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Independence lemma
Recurrence relations

\[ \psi_k = \left( \hat{P}_k \to 3 + P_k^{E \to 1} + \frac{P_0 \to 3}{2} \right)^2 \]

\[ F_k (3, 3, 2, 2 \to 3) = \psi_k \left( \frac{(1 - p_2) \cdot \hat{P}_k \to 3}{P_0 \to 1 + \frac{1}{2} p_2 P_k^{E \to 3} + (1 - p_2) \cdot \left( \frac{1}{2} P_k^{E \to 3} + \hat{P}_k \to 3 \right)} \right)^2 \]

\[ F_k (3, 3, 3 \to 3) = (\psi_k)^3 \]

\[ H_k (3, 3, 2, 2) = w_k^3 q_k^3 \left( q_k^2 \right)^2 \cdot \left( P_0 \to 1 + \frac{1}{2} p_2 P_k^{E \to 3} + (1 - p_2) \cdot \left( \frac{1}{2} P_k^{E \to 3} + \hat{P}_k \to 3 \right) \right)^2 \]

\[ H_k (3, 3, 3) = w_k^3 \left( q_k^3 \right)^3 \]

\[ w_{k+1}^1 = \left( \sum_{\ast} F_k(\ast \to 1) H_k(\ast) \right) / \left( \sum_{\ast, i} F_k(\ast \to i) H_k(\ast) \right) \]

\[ w_{k+1}^2 = \left( \sum_{\ast} F_k(\ast \to 2) H_k(\ast) \right) / \left( \sum_{\ast, i} F_k(\ast \to i) H_k(\ast) \right) \]

\[ w_{k+1}^3 = \left( \sum_{\ast} F_k(\ast \to 3) H_k(\ast) \right) / \left( \sum_{\ast, i} F_k(\ast \to i) H_k(\ast) \right) \]
Main results

Theorem
Every cubic graph with sufficiently large girth has the fractional chromatic number at most $2.2978$.

Corollary
Every cubic graph with sufficiently large girth contains an independent set of size $0.4352n$.

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Random cubic graph a.a.s contains an independent set of size $0.4352n$. 
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Thank you for your attention!