Computing Linkless and Flat Embeddings in $\mathbb{R}^3$

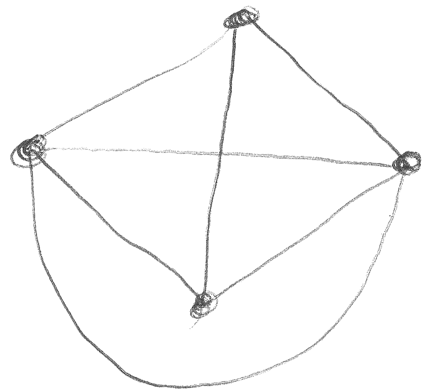
Stephan Kreutzer

joint work with
Ken-ichi Kawarabayashi
Bojan Mohar

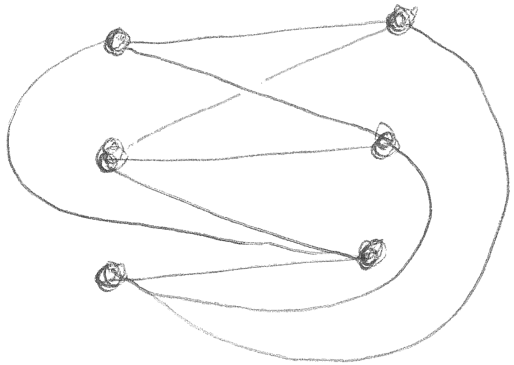
<u>Motivation:</u> Drawing graphs nicely.

<u>Usually:</u> Draw a graph on the plane.

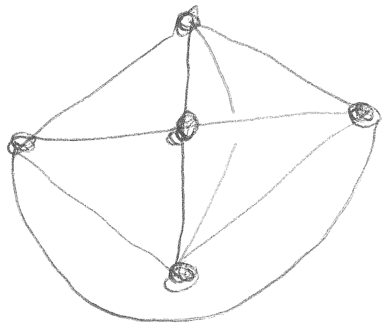
<u>planar graphs:</u> no crossing edges



Non-planar Graphs



$K_{3,3}$



K_5

Theorem: Every non-planar graph contains a subdivision of K_5 or $K_{3,3}$ as sub-graph.
(Kuratowski)

Def: Kuratowski graph: subdivision of $K_{3,3}$ or K_5.

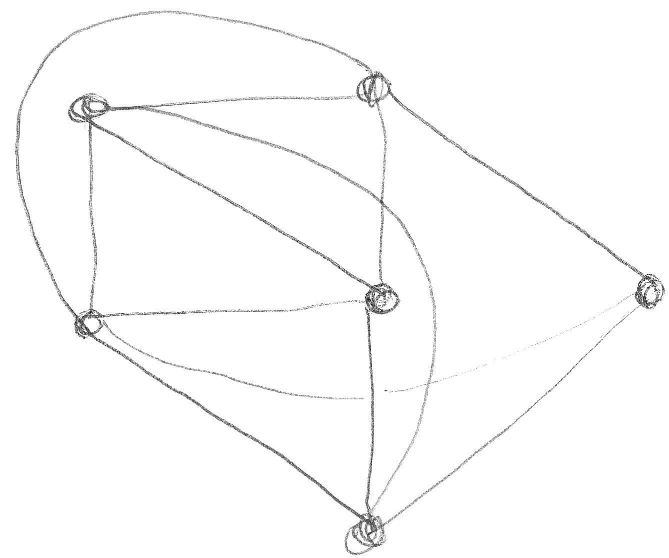
Topic of this talk:
Study nice drawings of graphs in $\mathbb{R}^3$

$\rightsquigarrow$ linkless or flat embeddings

<u>Def</u>: Let C_1, C_2 be two disjoint cycles embedded in $\mathbb{R}^3$

- C_1, C_2 are <u>unlinked</u> if there is a 2-dim top. disk in $\mathbb{R}^3$ containing one cycle but not the other.
- C_1, C_2 are <u>linked</u> otherwise

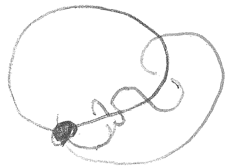
Graph G is linklessly embeddable if there is an embedding of G in $\mathbb{R}^3$ so that no two disjoint cycles are linked.



Flat Embeddings

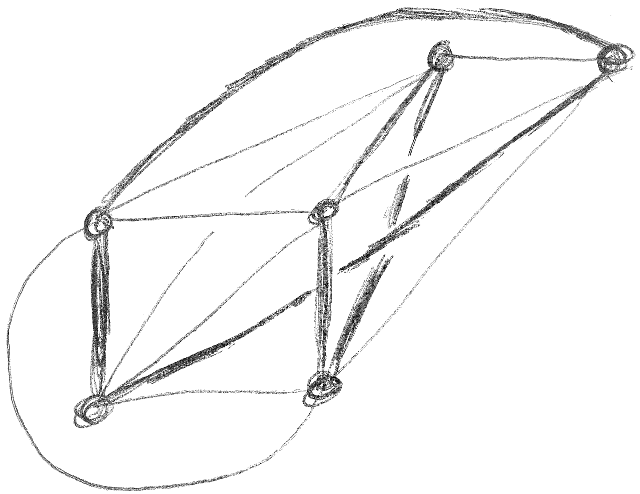
<u>Def</u>: An embedding of G in $\mathbb{R}^3$ is <u>flat</u> if for every cycle in G there is a closed top. disk D such that $D \cap G = \partial D = C$

<u>Obs</u>: Every flat embedding is linkless, the converse is false.



<u>Theorem</u>: (Robertson, Seymour, Thomas)
A graph is linklessly embeddable $\iff$ it has a flat embedding.

Not every graph is linklessly embeddable!

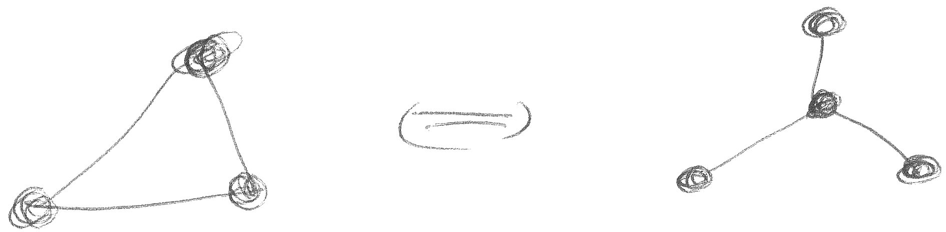


K_6

<u>Observation:</u> Class of linklessly embeddable graphs closed under minors.

$\Longrightarrow$ Characterisation of linklessly emb. graphs by a finite set of excluded minors (Rob, Sey.)

Δ-Y transformations:



<u>Def:</u> H and G are <u>$Y\Delta$ equivalent</u> if H can be obtained from G by a sequence of $Y\Delta$ transformations.

<u>Thm:</u> if H, G are $Y\Delta$ equivalent then
G linklessly emb. $\Longleftrightarrow$ H is linkl. emb.

(Rob, Sey, Tho)

Theorem: G linklessly emb. $\Longleftrightarrow$ it does not contain a graph of Petersen family as minor.

$\longrightarrow$ $O(n^3)$ decision alg. but no alg for computing linkless embeddings.

Thm: (van der Holst)
there is pol. time alg. which, given G either computes a linkless embedding or witnesses that there is none.

Main result: there is an $O(n^2)$ alg which, given G, decides if G has a linkless embedding and if so computes a flat embedding of G in $\mathbb{R}^3$.

Main result: there is an $O(n^3)$ alg which, given G, decides if G has a link. emb. and if so constructs a flat embedding.

High level proof idea:

1. Analyse the tree-width of G

a) tree-width high

reduce to a smaller graph

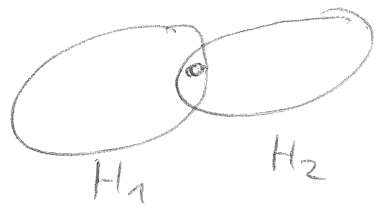
- we can delete a vertex
- recursively compute flat embedding
- put the vertex back in

b) tree-width small

- decompose into 4-conn. components
- use the fact that in 4-connected linklessly emb. graphs the Kuratowski subgraphs "commute"

(5) **Aim:** reduce the problem to 4-connected graphs

1- separations:

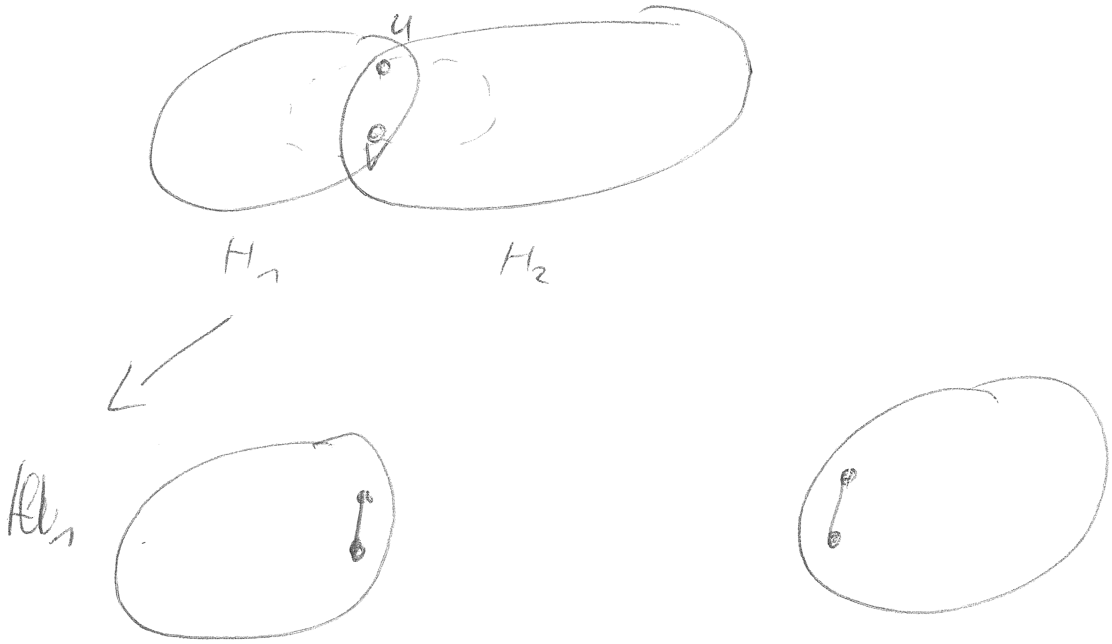


draw H_1 flat
H_2 flat

combine

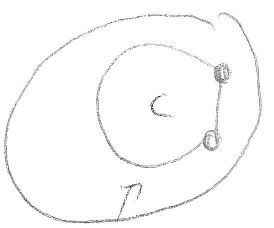
but no 1-separations

2- separations:



$H_1 + uv \leq 6$ $H_2 + uv \leq 6$

draw $H_1 + uv$ flat $H_2 + uv$ flat

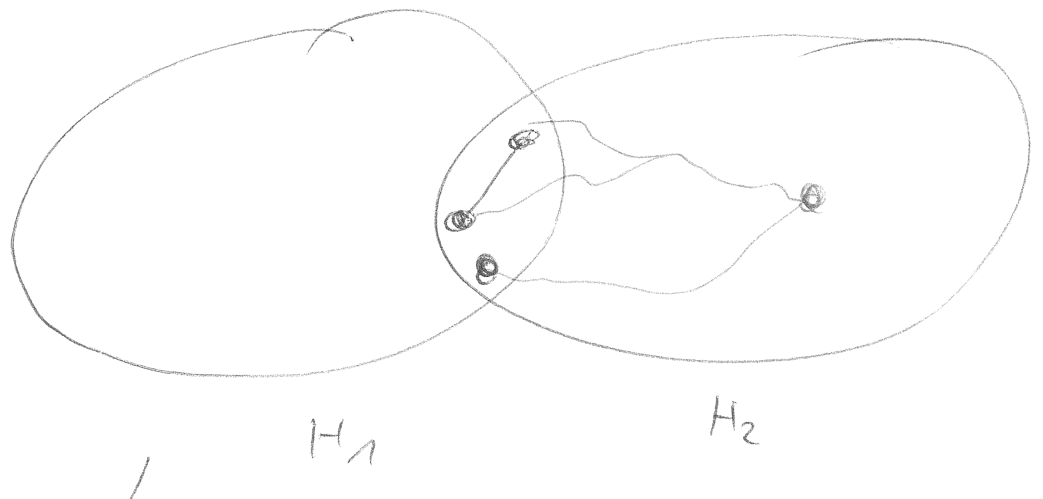


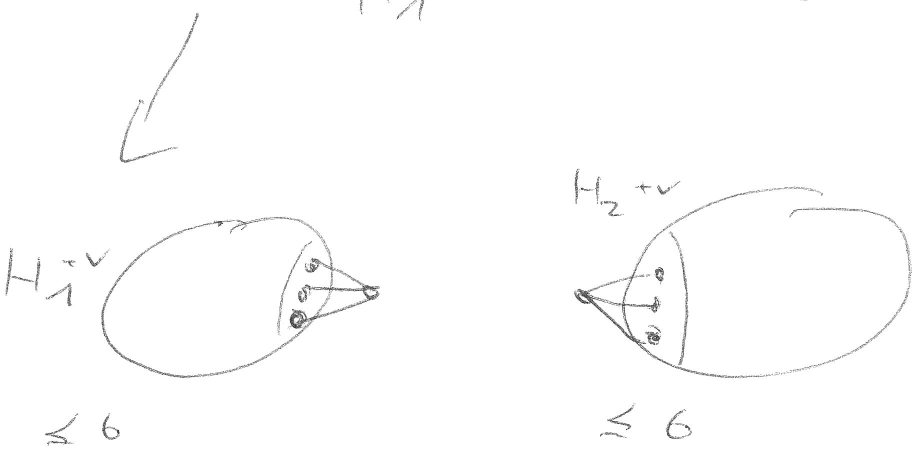
cycle $\Leftrightarrow$ there is disk with boundary C

embed $H_2 + uv$ there.

3-separations

no 2-sep.





perform $Y\Delta$ transform.

join at triangle.

$\Longrightarrow$ we can assume that G is 4-connected

Flat embeddings of 4-connected graphs

(R, S, T)

Lemma:

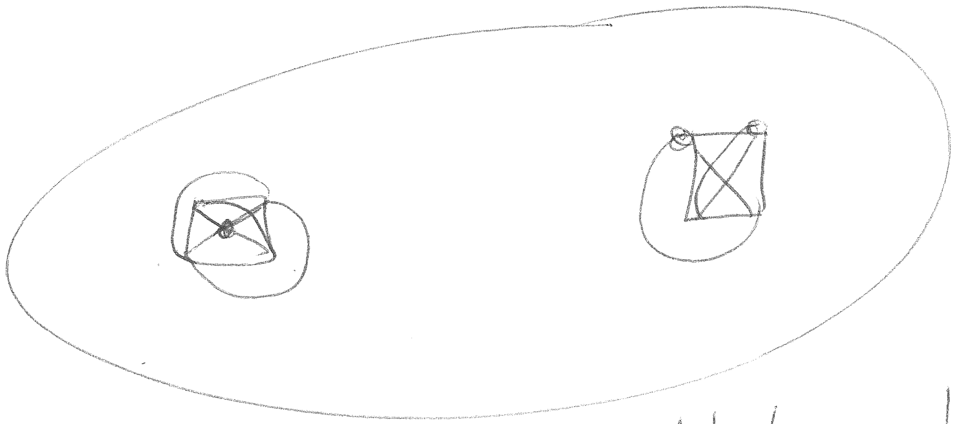
1) Any two flat embeddings of a planar graph are amb. isotopic.
2) K_5 and $K_{3,3}$ have exactly 2 non-amb. isotopic flat embeddgs.
3) Let ϕ_1, ϕ_2 be flat emb. of G which are not amb. isotopic.
Then there is a sub-graph $H \subseteq G$ isomorphic to a sub-division of $K_{3,3}$ or K_5 for which
$\phi_1|_H$ and $\phi_2|_H$ are not amb isotopic.

Def: ϕ_1, ϕ_2 are ambient isotopic ($\phi_1 \cong_{a.i.} \phi_2$) if there is an orientation preserving homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ mapping ϕ_1 to ϕ_2.

Note

consequence: if you draw all Kuratowski graphs carefully the rest attaches uniquely.

Problem: $K_5, K_{3,3}$ emb have ~~ko~~ k^2 new ai



how do you know that you have not made a mistake?