Computational complexity of homomorphism-homogeneity

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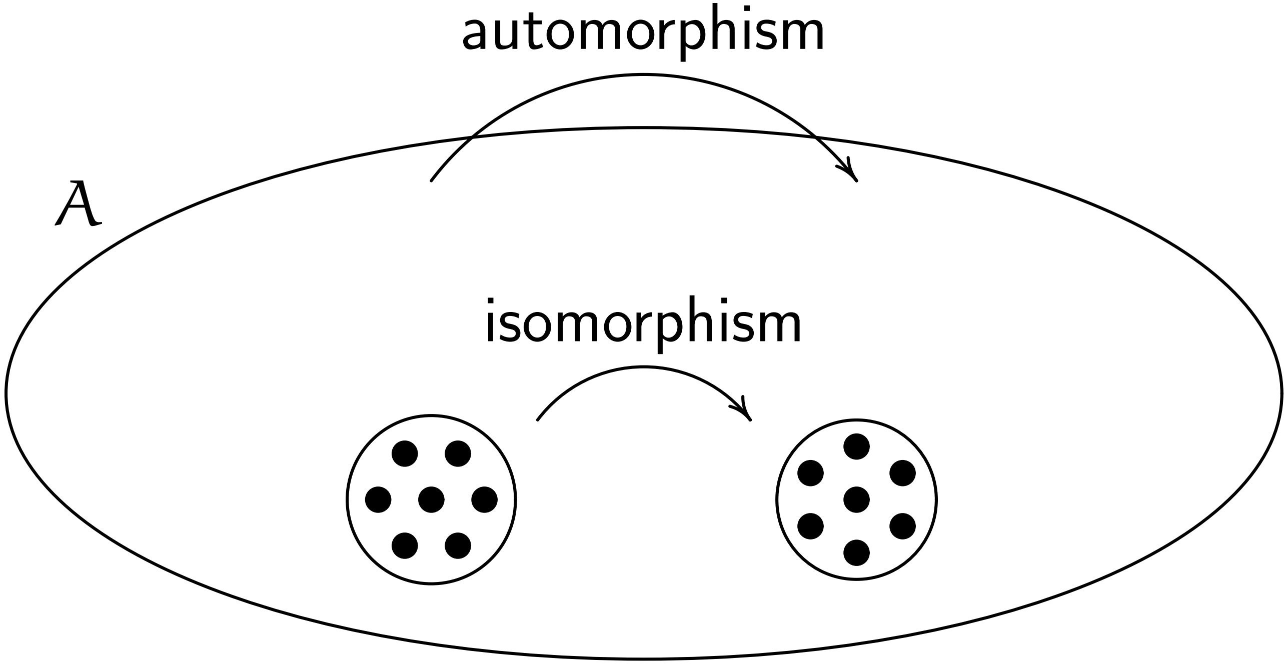
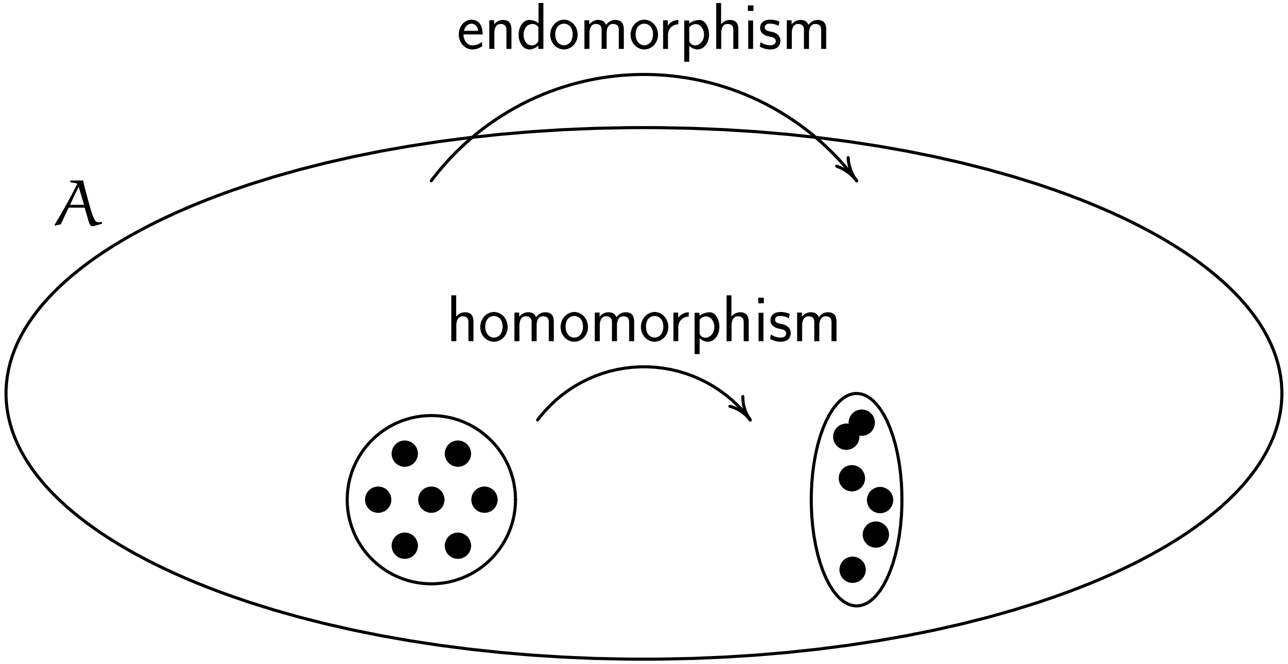
Homogeneous structures



Homomorphism-homogeneous structures



Cameron, P. J., Nešetřil, J., *Homomorphism-homogeneous relational structures*, Combinatorics, Probability and Computing 15, 91–103 (2006)

Homomorphism-homogeneous structures

2007. Mašulović: finite and infinite partially ordered sets
(Order 24(2007), 215–226)

2008. Ilić, Mašulović, Rajković: finite tournaments with loops
(Journal of Graph Theory 1(2008), 45–58)

2009. Mašulović: a class of point-line geometries (submitted)

2010. Cameron, Lockett: infinitely countable posets w.r.t. various types of morphisms (Discr Math 310(2010), 604-613)

2010. Mašulović, Pech: a model-theoretic approach in the fashion of Engeler, Ryll-Nardzewski, Svenonius
(to appear in Fund. Math.)

Homomorphism-homogeneous irreflexive structures

Theorem. [Nešetřil, Cameron 2006] *A finite graph* G *with no loops is homomorphism-homogeneous if and only if* $G \cong k \cdot K_n$ *for some* $k, n \geqslant 1$*.*

(Combinatorics, Probability and Computing 15(2006), 91–103)

Theorem. [($\ddot\smile$), Nenadov, Škorić] *Let* $D = (V, \rho)$ *be a finite irreflexive binary rel system. Then* D *is hom-hom if and only if*

1. $D \cong k \cdot K_n$ *for some* $k, n \geqslant 1$*, or*

2. $D \cong k \cdot C_3$ *for some* $k \geqslant 1$*.*

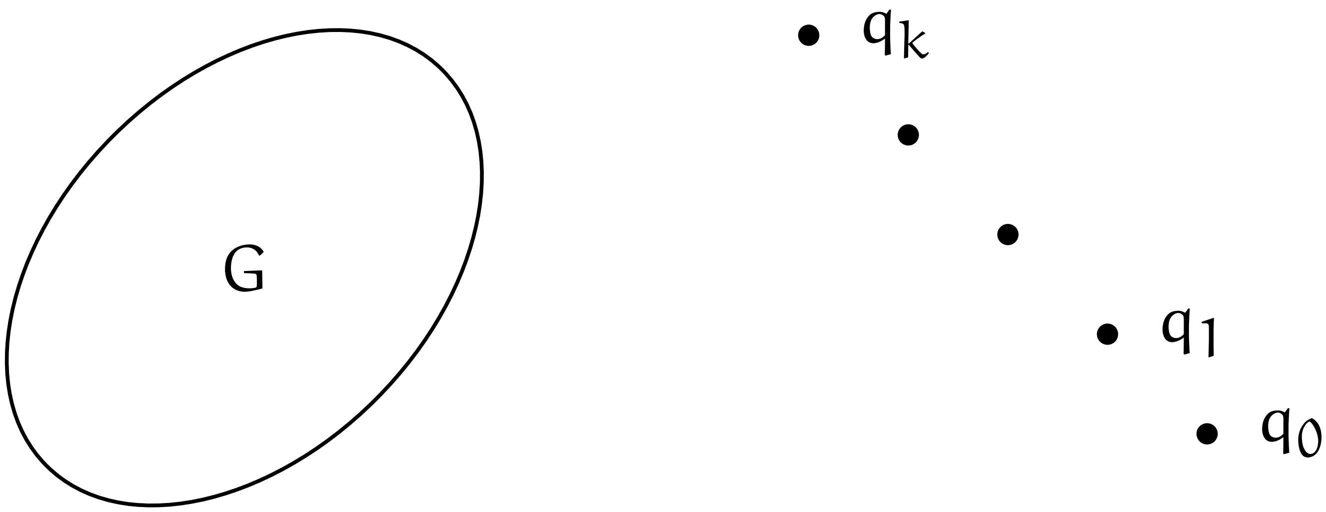
Homomorphism-homogeneous graphs with loops

What about graphs with loops?

2010. Rusinov, Schweitzer: *Homomorphism-homogeneous graphs*
(to appear in Journal of Graph Theory)

Theorem. [Rusinov, Schweitzer] *Deciding whether a finite graph with loops is hom-hom is a coNP-complete problem.*

Proof. [$(\ddot{\smile})$, Nenadov, Škorić]
Reduce the complement of the INDEPENDENT SET problem.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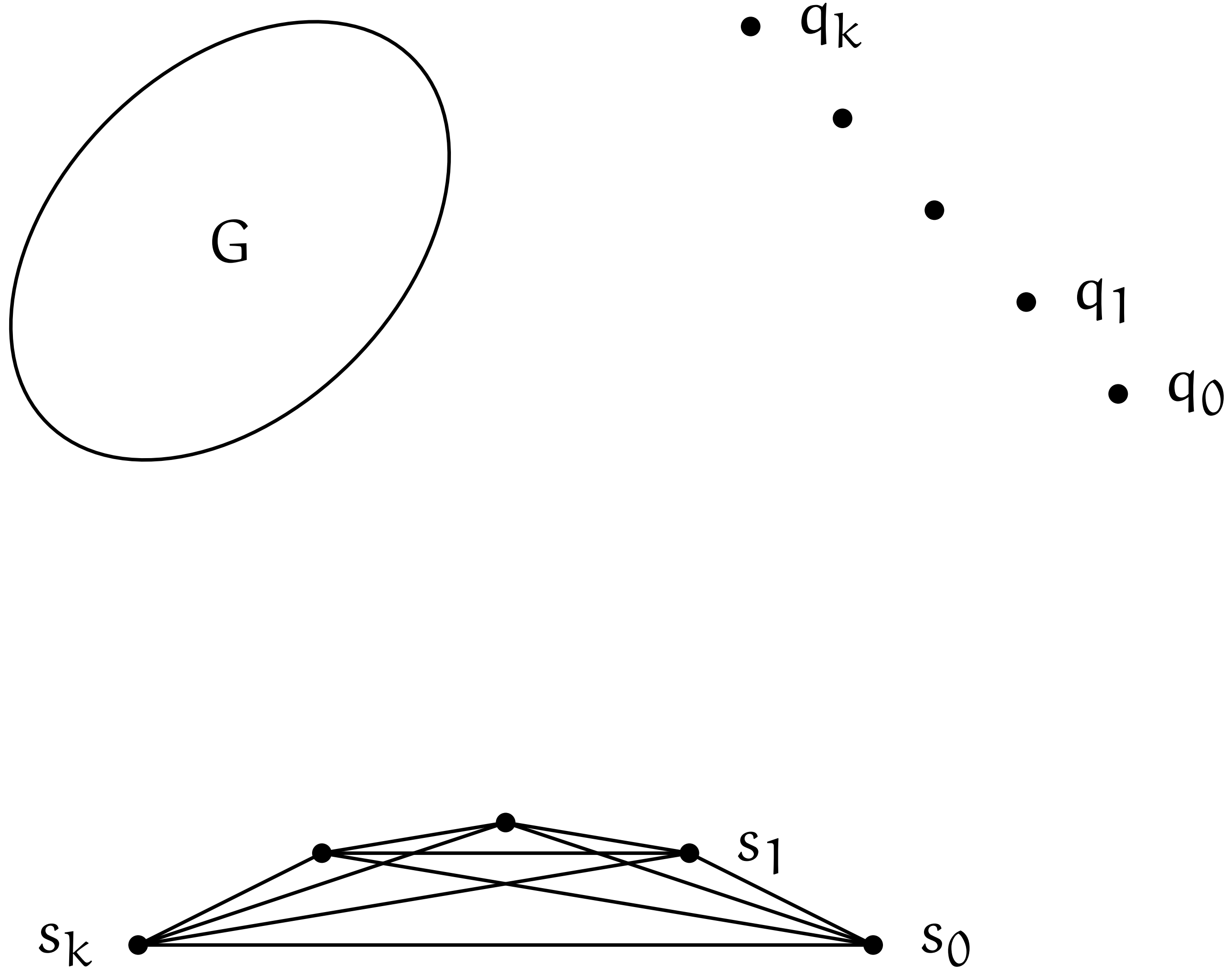
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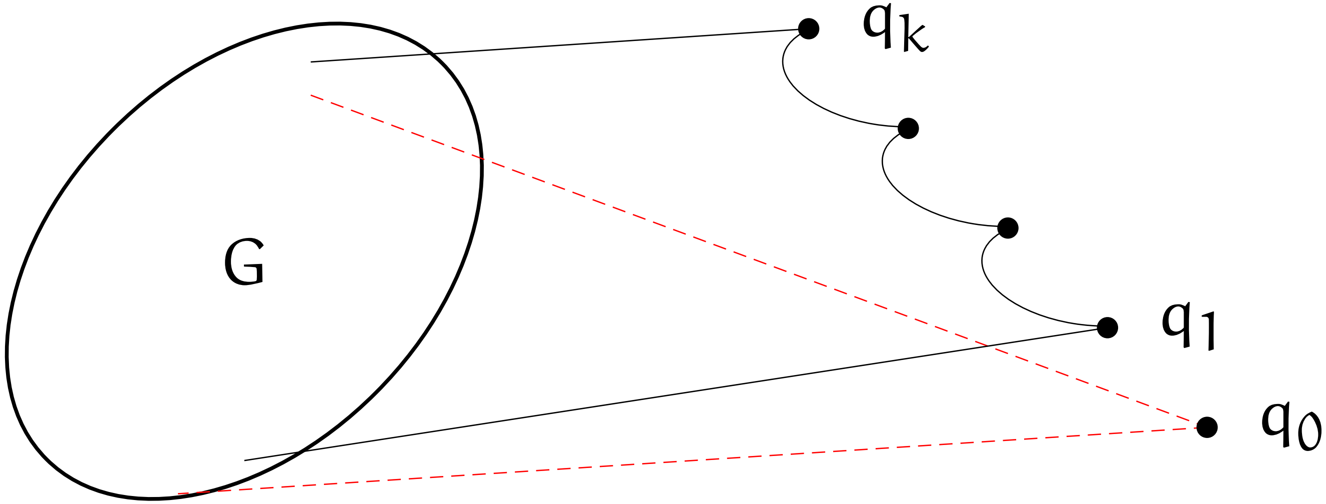
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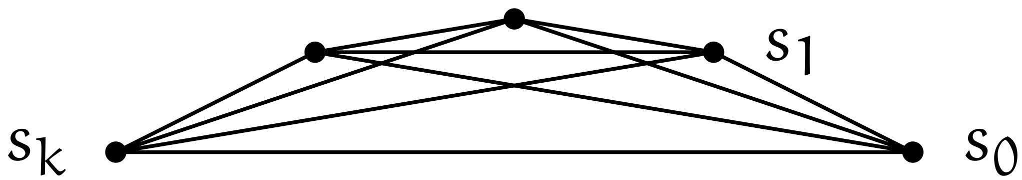
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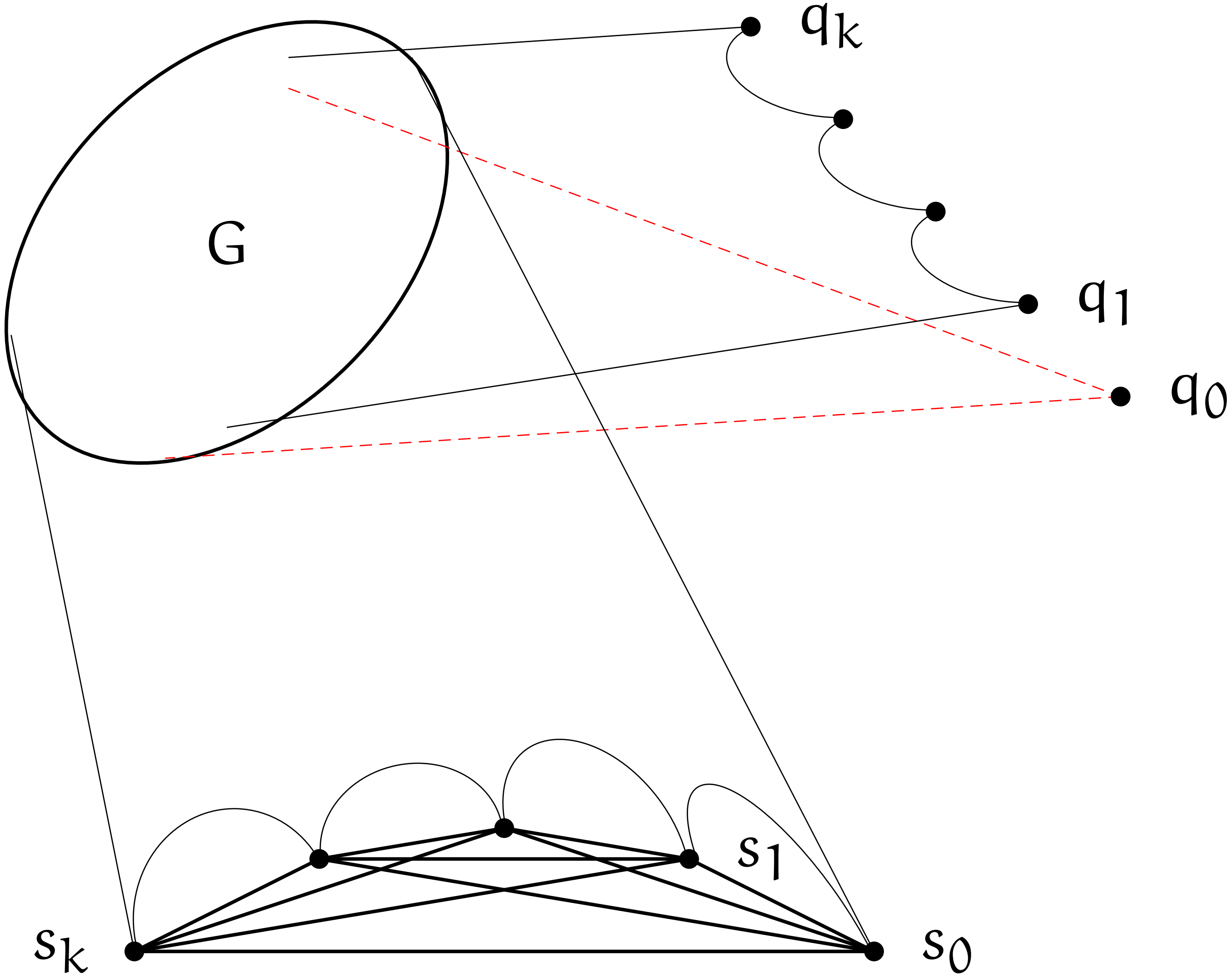

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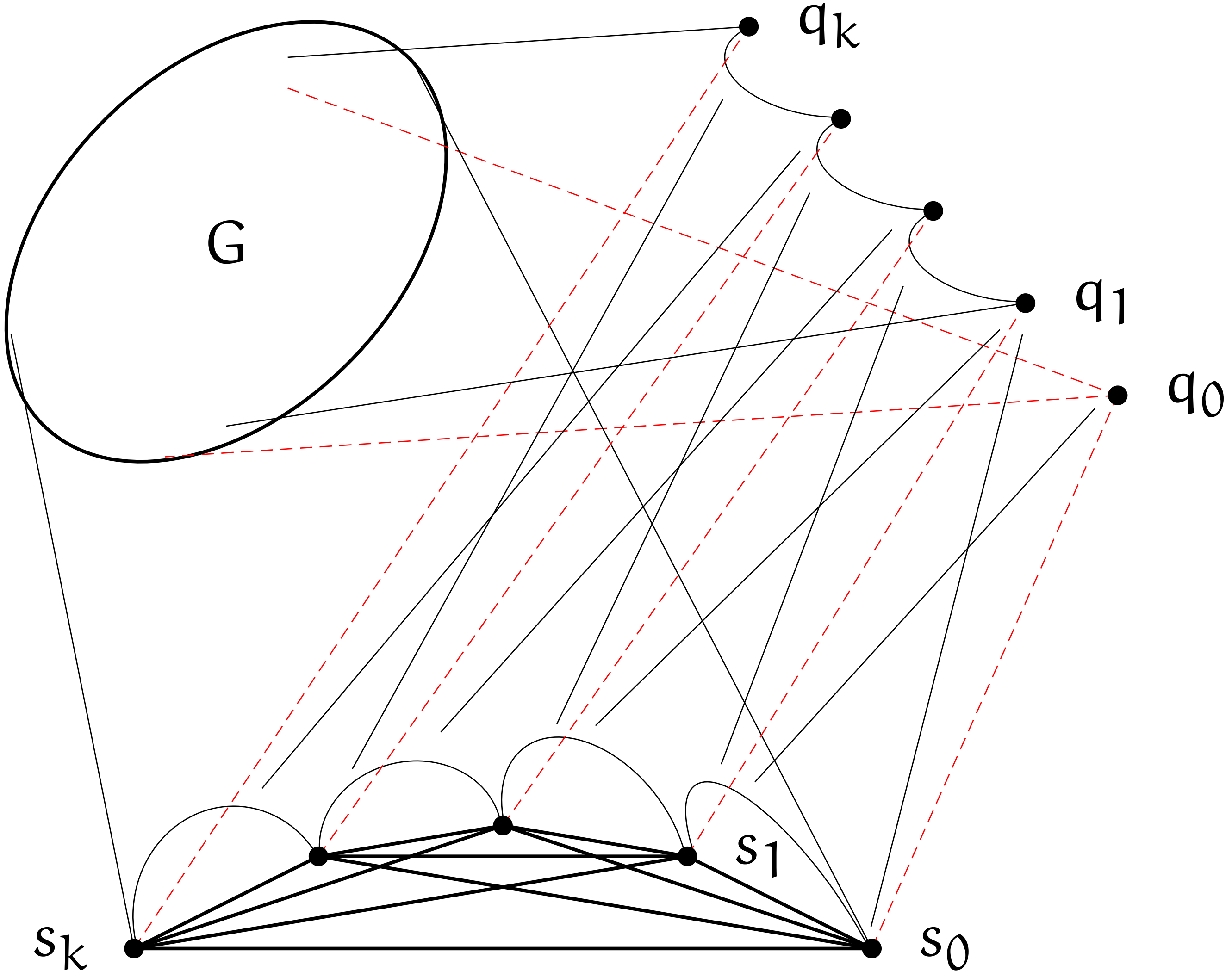

G
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One more coNP result

Theorem. $[(\ddot\smile)]$ *Deciding whether a finite metric space is hom-hom is a coNP-complete problem.*

Proof. Construct a functor $\mathsf{F} : \mathsf{FinGrph}^\circ \longrightarrow \mathsf{FinMetSpc}$ s.t.

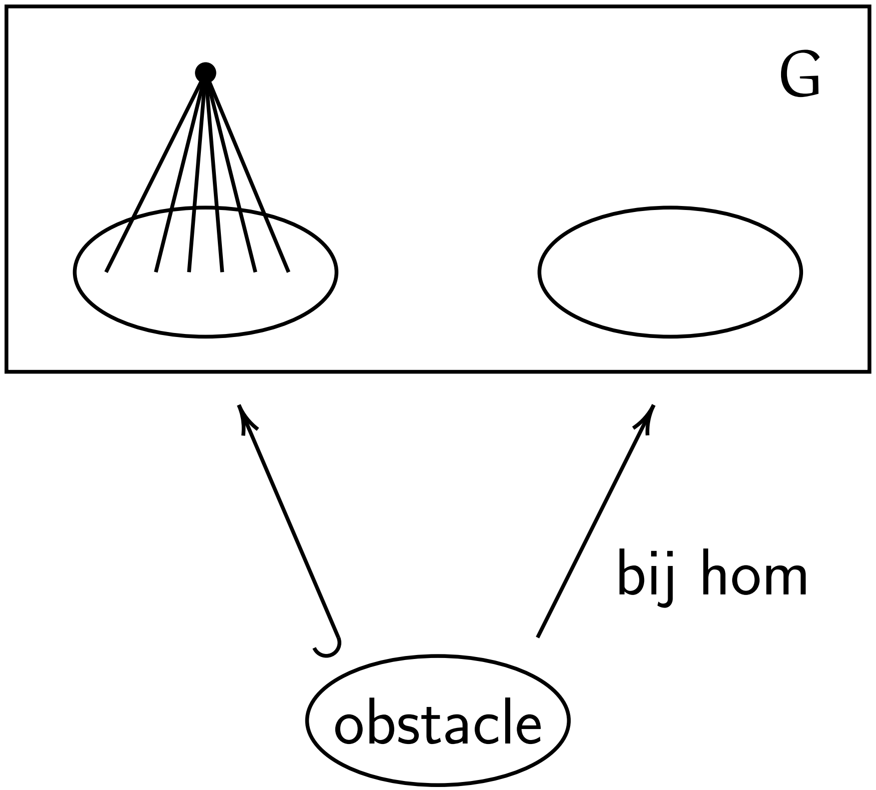
1) F is full and faithful
2) F preserves and creates embeddings
3) F is computable in polynomial time.

Then: G is hom-hom if and only if $\mathsf{F}(\mathsf{G})$ is hom-hom. $\square$

NB. [Dolinka, $(\ddot\smile)$] *Both the countable Urysohn's space* $\mathbb{U}$ *and its completion* $\mathbb{U}^*$ *are hom-hom.*

A dychotomy for graphs?

Rusinov, Schweitzer: If G is not hom-hom, then there exists an obstacle



A dychotomy for graphs?

Let $\mathcal{K}$ be a class of graphs and let $\mathcal{O}$ be the set of all obstacles that appear in $\mathcal{K}$ (up to isomorphism).

Fact. *If $\mathcal{O}$ is finite then deciding whether a graph from $\mathcal{K}$ is hom-hom is in P.*

Hypothesis. *If $\mathcal{O}$ is infinite then deciding whether a graph from $\mathcal{K}$ is hom-hom is coNP-complete.*

Homomorphism-homogeneous algebraic systems

An algebra A is *quasi-injective* if every hom $f : S \to A$ from a subalgebra S of A into A extends to an endomorphism of A.

If A is *finite*: A is quasi-inj iff A is hom-hom.

Finite quasi-inj ($=$ hom-hom) groups: Bertholf, Walls 1979

Homomorphism-homogeneous algebraic systems

$\Delta = (r_1, r_2, \ldots, r_n)$ is a finite alg type; $r_1, r_2, \ldots, r_n \geqslant 0$

$\mathsf{FinAlg}(\Delta)$ is the category of all finite algebras of type Δ

Problem. Given Δ, find computational complexity of deciding hom-hom in $\mathsf{FinAlg}(\Delta)$.

Lemma. *Deciding hom-hom in* $\mathsf{FinAlg}(\Delta)$ *is in coNP.*

FinAlg(1) – **monounary algebras**

(A, f), where $f : A \to A$

A connected component $S \subseteq A$ is *regular* if

- the set C of cyclic elements of S is nonempty, and
- either every branch starting at a cyclic element is infinite, or every branch starting at a cyclic element is finite and $\mathrm{ht}(a) = \mathrm{ht}(b)$ for all leaves a, b in S.

FinAlg(1) – **monounary algebras**

Theorem. [Jungábel, (☺)] *Let* (A, f) *be a monounary alg. Then* (A, f) *is hom-hom iff*

1. *every branch in* (A, f) *is infinite; or*
2. *every connected component in* (A, f) *is regular, and for any two connected components* $S_1, S_2 \subseteq A$*, if* $\mathrm{cn}(S_1) | \mathrm{cn}(S_2)$ *then* $\mathrm{ht}(S_1) \geqslant \mathrm{ht}(S_2)$.

Algebras with at least one at least binary operation

Theorem. *Deciding hom-hom in* $\mathsf{FinAlg}(2)$ *is coNP-complete.*

Proof. Construct a functor $\mathbb{F}: \mathsf{FinGrph}^\circ \to \mathsf{FinAlg}(2)$ s.t.

1) $\mathbb{F}$ is faithful and “almost” full

2) $\mathbb{F}$ preserves embeddings and creates “almost all” embeddings

3) $\mathbb{F}$ is computable in polynomial time.

Then: $\mathbb{G}$ is hom-hom if and only if $\mathbb{F}(\mathbb{G})$ is hom-hom. $\square$

A dychotomy for algebras?

Corollary. *Deciding hom-hom in* FinAlg(1) *is in P.*

Corollary. *Deciding hom-hom in* FinAlg(Δ)*, where* Δ *contains at least one at least binary operation, is coNP-complete.*

Open problem. What about FinAlg$(1, 1, \ldots, 1)$ and FinAlg$(1, 1, \ldots, 1, 0, 0, \ldots, 0)$?

Homomorphism-homogeneous lattices $(\mathrm{L}, \wedge, \vee)$

Theorem. [Dolinka, $(\ddot{\smile})$] *A lattice* L *is homomorphism-homogeneous if and only if it is either a chain, or every interval of* L *is a boolean lattice.*

(Cf. *Every lattice* $(\mathrm{L}, \leqslant)$ *is homomorphism-homogeneous*)

Corollary. [Dolinka, $(\ddot{\smile})$] *A finite lattice* L *is hom-hom if and only if it is either a chain, or a direct power of* $0 < 1$.

(*Remarks on homomorphism-homogeneous lattices and semilattices*; to appear in Monatshefte für Mathematik)

Homomorphism-homogeneous semilattices $(S, \wedge)$

Lemma [Dolinka, $(\ddot\smile)$] *Let S be a hom-hom semilattice. Then S is either a tree, or it is locally bounded.*

Consequently, if S is a finite hom-hom semilattice, then it is either a tree, or the $\wedge$-semilattice reduct of a lattice.

Theorem. [Dolinka, $(\ddot\smile)$] *Every tree is a hom-hom semilattice.*

Theorem. [Dolinka, $(\ddot\smile)$] *Let $(\mathrm{L}, \wedge, \vee)$ be a distributive lattice. Then $(\mathrm{L}, \wedge)$ is a hom-hom semilattice.*

(*Remarks on homomorphism-homogeneous lattices and semilattices*; to appear in Monatshefte für Mathematik)

Homomorphism-homogeneous semilattices $(S, \wedge)$

Lemma [Dolinka, ($\ddot{\smile}$)] $(\mathrm{N}_5, \wedge)$ *and* $(\mathrm{M}_3, \wedge)$ *are homomorphism-homogeneous.*

Open problem. Characterize hom-hom semilattices.

(*Remarks on homomorphism-homogeneous lattices and semilattices*; to appear in Monatshefte für Mathematik)

NB. [Dolinka] *The universal homogeneous countable semilattice is hom-hom.*

Thank you