On the Degree/Diameter Problem for Graphs Embedded on Surfaces

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Degree/Diameter Problem

Problem (The Degree/Diameter Problem)

Given a class $\mathcal{C}$ of graphs and integers Δ and k, determine the maximum number $N(\Delta, k, \mathcal{C})$ of vertices in a graph (in $\mathcal{C}$) having maximum degree Δ and diameter k.

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Examples of classes:

- $\mathcal{G}$ of all graphs.
- Bipartite graphs.
- Vertex-transitive graphs.
- Cayley graphs.
- $\mathcal{P}$ **of planar graphs.**
- $\mathcal{G}_\Sigma$ **of graphs embeddable in a given a surface Σ.**

DDP Overview – Bounds

Moore Bound:

$$N(\Delta, k, \mathcal{G}) \leq M(\Delta, k) = 1 + \Delta + \Delta(\Delta - 1) + \ldots + \Delta(\Delta - 1)^{k-1}$$

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(Bannai and Ito 1973 / Damerrell 1973)

$N(\Delta, k, \mathcal{G}) \leq M(\Delta, k) - 1$ *for almost all* (Δ, k) *pairs.*

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$N(\Delta, k, \mathcal{G}) \leq M(\Delta, k) - 2$ *for almost all* (Δ, k) *pairs.*

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(FP and G. Pineda-Villavicencio 2012)

$N(\Delta, k, \mathcal{G}) \leq M(\Delta, k) - 3$ *for at least* $3/4$ *of all* (Δ, k) *pairs.*

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Lower Bound:

$$N(\Delta, k, \mathcal{G}) \geq (\Delta/1.59)^k.$$

DDP Overview – Table

Δ\k	2	3	4	5	6	7	8	9	10
3	10	20	38	70	132	196	336	600	1,250
4	15	41	98	364	740	1,320	3,243	7,575	17,703
5	24	72	212	624	2,772	5,516	17,030	57,840	187,056
6	32	111	390	1,404	7,917	19,383	76,461	307,845	1,253,615
7	50	168	672	2,756	11,988	52,768	249,660	1,223,050	6,007,230
8	57	253	1,100	5,060	39,672	131,137	734,820	4,243,100	24,897,161
9	74	585	1,550	8,200	75,893	279,616	1,686,600	12,123,288	65,866,350
10	91	650	2,286	1,340	134,690	583,083	4,293,452	27,997,191	201,038,992
11	104	715	3,200	19,500	156,864	1,001,268	7,442,328	72,933,102	600,380,000
12	133	786	4,680	29,470	359,772	1,999,500	15,924,326	158,158,875	1,506,252,500
13	162	851	6,560	40,260	531,440	3,322,080	29,927,790	249,155,760	3,077,200,700
14	183	916	8,200	57,837	816,294	6,200,460	55,913,932	600,123,780	7,041,746,081
15	186	1,215	11,712	76,518	1,417,248	8,599,986	90,001,236	1,171,998,164	10,012,349,898
16	198	1,600	14,640	132,496	1,771,560	14,882,658	140,559,416	2,025,125,476	12,951,451,931

Diagrams Approach (Fellows, Hell, Seyffarth 1998)

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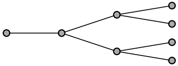
- A (Δ, γ)-**tree** is a tree of depht γ with its root and leaves having degree 1 and all other vertices having degree Δ.



A $(3, 3)$-tree

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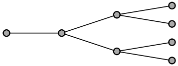


A $(3, 3)$-tree

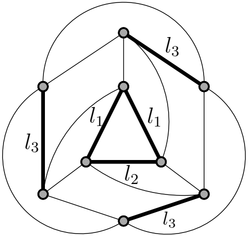
- A (Δ, β)-**pod** is the planar graph obtained from two $(\Delta, \lfloor \beta/2 \rfloor)$-trees, identifying their leaves if β is even and matching their leaves if β is odd.

A $(3, 6)$-pod and a $(3, 7)$-pod

Diagrams Approach (Fellows, Hell, Seyffarth 1998)

Definition (Diagram)

A *diagram* $\mathcal{D}$ is a multigraph, where **some** edges are labelled in the form $\alpha(\Delta, \beta)$.



$$l_1 = \lfloor \tfrac{\Delta-2}{2} \rfloor (\Delta, k-1)$$
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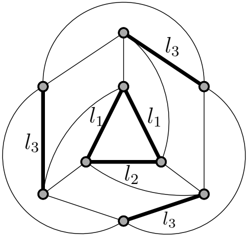

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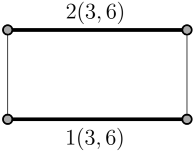
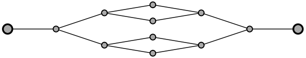
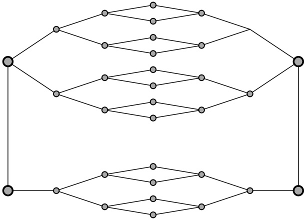
- A labelled edge is called **thick**, otherwise it is called **thin**.
- A vertex incident to a thick edge is called **thick**, otherwise it is called **thin**.

Given a diagram $\mathcal{D}$, the graph $G(\mathcal{D})$ is obtained by replacing thick edges by α "disjoint" (Δ, β)-pods.







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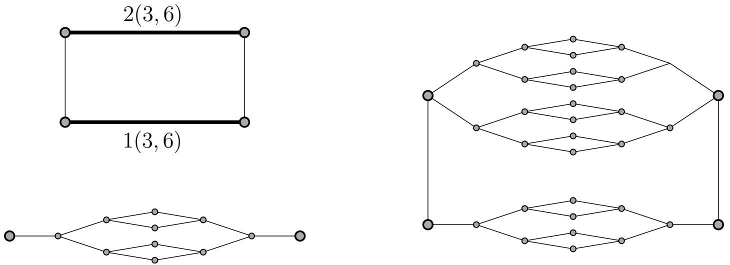

Observation

If $\mathcal{D}$ is planar then $G(\mathcal{D})$ is also planar.

Diagrams Approach (Fellows, Hell, Seyffarth 1998)

Let $\mathcal{D}^k_\Delta$ denote any diagram $\mathcal{D}$ with maximum (labelled) degree Δ and labels $\alpha(\Delta, \beta)$ satisfying: $\beta \leq k$.
Let the **weight** of a walk in $\mathcal{D}^k_\Delta$ be the sum of all the β values on the labels of the edges of the walk.

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Proposition (Fellows, Hell and Seyfart 1998)

Suppose that for a diagram $\mathcal{D}^k_\Delta$ the following holds:

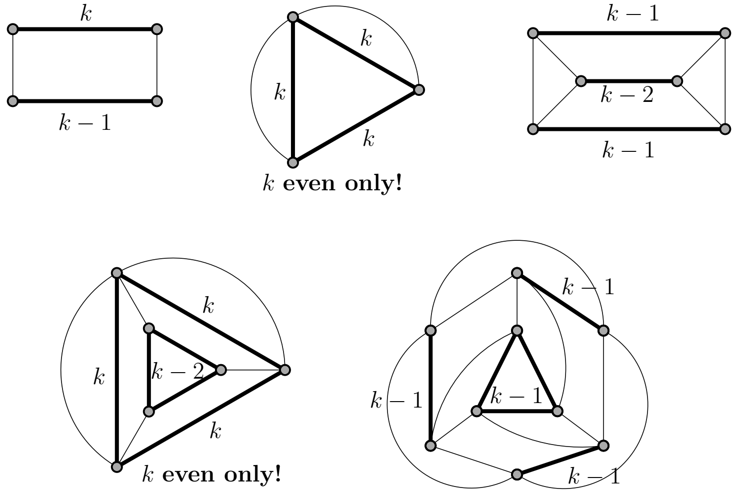
(i) *Any two thick edges of $\mathcal{D}^k_\Delta$ are contained in a closed walk of weight at most $2k + 1$.*

(ii) *For any thin vertex v and any thick edge e of $\mathcal{D}^k_\Delta$, v and e lie in closed walk of weight at most $2k + 1$.*

(iii) *There is a path of weight at most k between any two thin vertices of $\mathcal{D}^k_\Delta$.*

Then the graph $G(\mathcal{D}^k_\Delta)$ has diameter at most k.

Diagrams Approach (Fellows, Hell, Seyffarth 1998)



Planar Graphs – Table

Δ \ k	2	3	4	5	6	7	8	9	10
3	*FHS* **7**	**12**	18	28	*E* 38	*FHS* 53	*FHS* 77	*FHS* 109	*FHS* 157
4	*YLD* **9**	16	27	*FHS* 44	81	*FHS* 134	*T* 243	*FHS* 404	*FHS* 728
5	*YLD* **10**	*FHS* 19	*E* 39	*FHS* 73	*T* 158	*FHS* 289	*T* 638	*FHS* 1 153	*T* 2 558
6	*YLD* **11**	*FHS* 24	*T* 55	*FHS* 114	*T* 280	*FHS* 564	*T* 1 405	*FHS* 2 814	*T* 7 030
7	*YLD* **12**	*FHS* 28	*T* 74	*FHS* 161	*T* 452	*FHS* 959	*T* 2 720	*FHS* 5 747	*T* 16 328
8	*FHS* **13**	*FHS* 33	*T* 97	*FHS* 225	*T* 685	*FHS* 1 569	*T* 4 901	*FHS* 10 977	*T* 33 613
9	*FHS* **14**	*FHS* 37	*T* 122	*FHS* 289	*T* 986	*FHS* 2 305	*T* 7 898	*FHS* 18 433	*T* 63 194
10	*FHS* **16**	*FHS* 42	*T* 151	*FHS* 372	*T* 1 366	*FHS* 3 342	*T* 12 301	*FHS* 30 072	*T* 110 716

Planar Graphs – Improvements

Proposition

Suppose that for a diagram $\mathcal{D}^k_\Delta$ the following holds:

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(ii) ...

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Then the graph $G(\mathcal{D}^k_\Delta)$ has diameter at most k.

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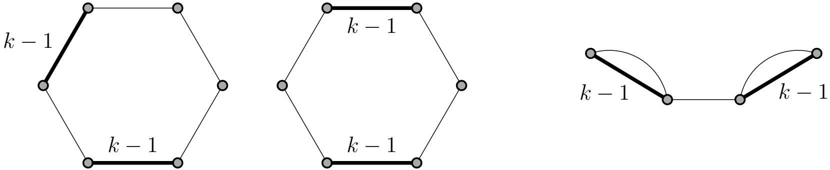

Planar Graphs – Improvements

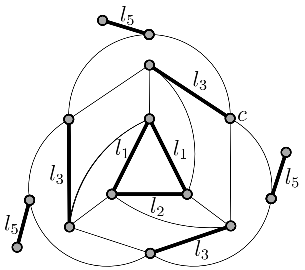

Diagram Z_Δ^k (even Δ)

(a)

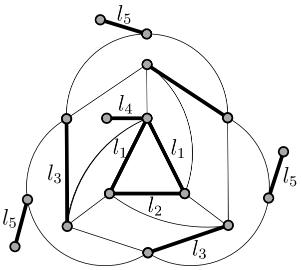

Diagram Z_Δ^k (odd Δ)

(b)

$l_1 = \lfloor\frac{\Delta-2}{2}\rfloor(\Delta, k-1)$

$l_2 = \lceil\frac{\Delta-2}{2}\rceil(\Delta, k-1)$

$l_3 = (\Delta-3)(\Delta, k-1)$

$l_4 = 1(\Delta, \frac{k-3}{2})$

$l_5 = (\Delta-2)(\Delta, \lfloor\frac{k-4}{2}\rfloor)$

Planar Graphs – Table (updated)

Δ \ k	2	3	4	5	6	7	8	9	10
3	*FHS* **7**	**12**	18	28	*E* 38	*FHS* 53	*FHS* 77	*FHS* 109	*FHS* 157
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Toroidal Graphs (odd k)

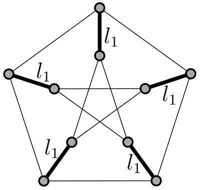

Diagram P_Δ^k

(a)

$$l_1 = (\Delta - 2)(\Delta, k-1)$$
$$l_2 = (\Delta - 4)(\Delta, k-1)$$

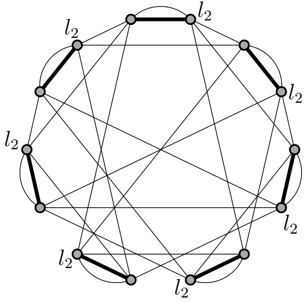

Diagram Q_Δ^k

(b)

Toroidal Graphs – Table (started)

Δ \ k	2	3	4	5	6	7	8	9	10
3	$\underline{r}$ 10	$\underline{r}$ 16	$\underline{r}$ 26	$\underline{r}$ 38	$\underline{r}$ 56	$\underline{r}$ 74	$\underline{r}$ 92	P 120	P 160
4	$\underline{r}$ 13	$\underline{r}$ 25	$\underline{r}$ 41	$\underline{r}$ 61	P 90	P 180	P 270	P 540	P 810
5	$\underline{r}$ 16	$\underline{r}$ 30	$\underline{r}$ 48	P 100	P 160	P 400	P 640	P 1 600	P 2 560
6	$\underline{r}$ 19	$\underline{r}$ 37	$\underline{r}$ 61	P 150	$\underline{p}$ 280	P 750	$\underline{p}$ 1 405	P 3 750	$\underline{p}$ 7 030
7	$\underline{p}$ 12	P 35	$\underline{p}$ 74	P 210	$\underline{p}$ 452	P 1 260	$\underline{p}$ 2 720	P 7 560	$\underline{p}$ 16 328
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Graphs on a Surface Σ (odd k)

Theorem (Map Colouring Theorem; Heawood, Ringels and Young)

Let Σ be a surface with Euler genus g and let G be a graph embedded in Σ. Then

$$\chi(G) \leq \frac{7 + \sqrt{1 + 24g}}{2}.$$

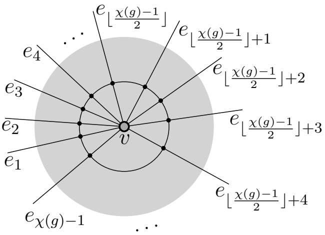
Furthermore, with the exception of the Klein bottle where $\chi(G) \leq 6$, there is a complete graph G embedded in Σ realising the equality.

Define the chromatic number of a surface Σ as

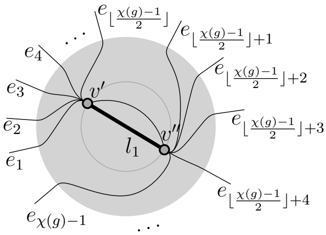
$$\chi(\Sigma) = \frac{7 + \sqrt{1 + 24g}}{2}$$

.

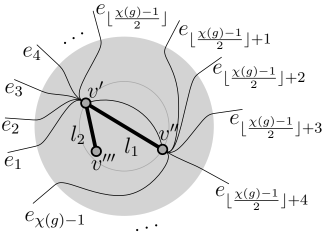
Graphs on a Surface Σ (odd k) – Vertex Splitting



(a)



(b)



(c)

$l_1 = (\Delta - 1 - \lceil \frac{\chi(g)-1}{2} \rceil)(\Delta, k-1)$

$l_2 = 1(\Delta, \frac{k-3}{2})$

Neighbourhood $B_\epsilon(v)$

$$l_1 = (\Delta - 4)(\Delta, k - 1)$$

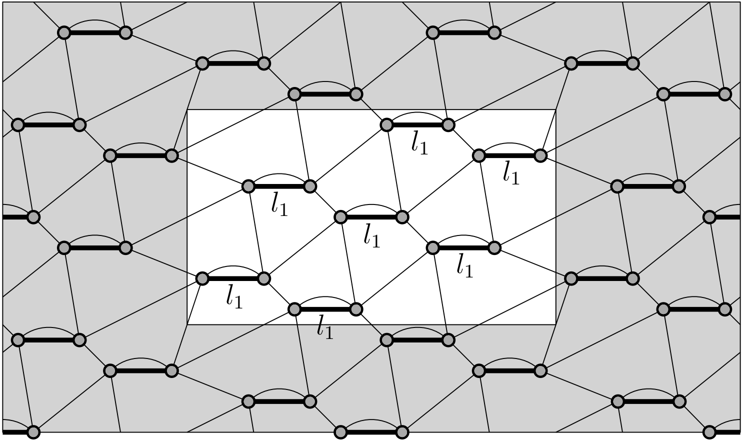


Diagram Q^k_Δ

Theorem (FP and G. Pineda-Villavicencio 2013)

For every surface Σ *of Euler genus* g*, and for every* $\Delta > \lceil \frac{\chi(\Sigma)-1}{2} \rceil + 1$ *and every* **odd** $k \geq 3$, $N(\Delta, k, \Sigma) \geq$

$$\chi(\Sigma)\left(\Delta - 1 - \left\lceil \frac{\chi(\Sigma)-1}{2} \right\rceil\right) \frac{\Delta(\Delta-1)^{\frac{k-3}{2}} - 2}{\Delta - 2} + 2\chi(\Sigma).$$

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Conjecture (FP and G. Pineda-Villavicencio 2013)

For each surface Σ and each **even** $k \geq 2$, *there exists a constant $\Delta_{(\Sigma,k)}$ such that for every degree $\Delta \geq \Delta_{(\Sigma,k)}$*

$$N(\Delta, k, \Sigma) = N(\Delta, k, \mathcal{P}).$$

Graphs on a Surface Σ – Bounds

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True for $k = 2$ (Knor and Širáň 1997).

Thanks!