Domination Game

Sandi Klavžar

Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

Institute of Mathematics, Physics and Mechanics, Ljubljana

Midsummer Combinatorial Workshop XIX
July 29 - August 2, 2013, Prague

The game

Domination game on G

- For a graph $G = (V, E)$, the domination number of G is the minimum number, denoted $\gamma(G)$, of vertices in a subset A of V such that $V = N[A] = \cup_{x \in A} N[x]$.

Domination game on G

- For a graph $G = (V, E)$, the domination number of G is the minimum number, denoted $\gamma(G)$, of vertices in a subset A of V such that $V = N[A] = \cup_{x \in A} N[x]$.

Two players, Dominator ($\mathcal{D}$) and Staller ($\mathcal{S}$), take turns choosing a vertex in G.

Domination game on G

- For a graph $G = (V, E)$, the domination number of G is the minimum number, denoted $\gamma(G)$, of vertices in a subset A of V such that $V = N[A] = \cup_{x \in A} N[x]$.

Two players, Dominator ($\mathcal{D}$) and Staller ($\mathcal{S}$), take turns choosing a vertex in G.

- If C denotes the set of vertices chosen at some point in a game and $\mathcal{D}$ or $\mathcal{S}$ chooses vertex w, then $N[w] - N[C] \neq \emptyset$.

Domination game on G

- For a graph $G = (V, E)$, the domination number of G is the minimum number, denoted $\gamma(G)$, of vertices in a subset A of V such that $V = N[A] = \cup_{x \in A} N[x]$.

Two players, Dominator ($\mathcal{D}$) and Staller ($\mathcal{S}$), take turns choosing a vertex in G.

- If C denotes the set of vertices chosen at some point in a game and $\mathcal{D}$ or $\mathcal{S}$ chooses vertex w, then $N[w] - N[C] \neq \emptyset$.
- $\mathcal{D}$ uses a strategy to end the game in as few moves as possible; $\mathcal{S}$ uses a strategy that will require the most moves before the game ends.

Game domination number

- The game domination number of G is the number of moves, $\gamma_g(G)$, when $\mathcal{D}$ moves first and both players use an optimal strategy. (Game 1)

Game domination number

- The game domination number of G is the number of moves, $\gamma_g(G)$, when $\mathcal{D}$ moves first and both players use an optimal strategy. (Game 1)
- $d_1, s_1, d_2, s_2, \ldots$ denotes the sequence of moves.

Game domination number

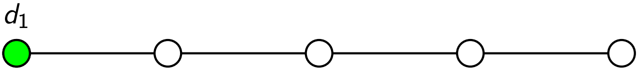
- The game domination number of G is the number of moves, $\gamma_g(G)$, when $\mathcal{D}$ moves first and both players use an optimal strategy. (Game 1)
- $d_1, s_1, d_2, s_2, \ldots$ denotes the sequence of moves.
- The staller-start game domination number of G is the number of moves, $\gamma_g'(G)$, when $\mathcal{S}$ moves first and both players use an optimal strategy. (Game 2)

Game domination number

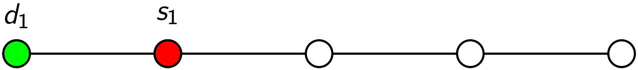
- The game domination number of G is the number of moves, $\gamma_g(G)$, when $\mathcal{D}$ moves first and both players use an optimal strategy. (Game 1)
- $d_1, s_1, d_2, s_2, \ldots$ denotes the sequence of moves.
- The staller-start game domination number of G is the number of moves, $\gamma_g'(G)$, when $\mathcal{S}$ moves first and both players use an optimal strategy. (Game 2)
- $s_1', d_1', s_2', d_2', \ldots$ denotes the sequence of moves.

The game on P_5

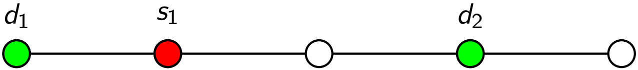
The game on P_5



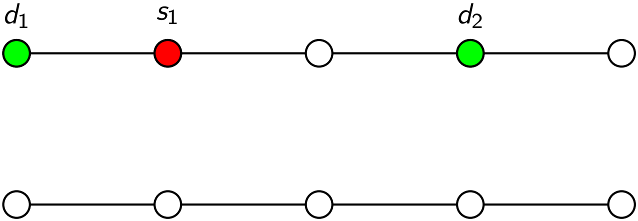
The game on P_5



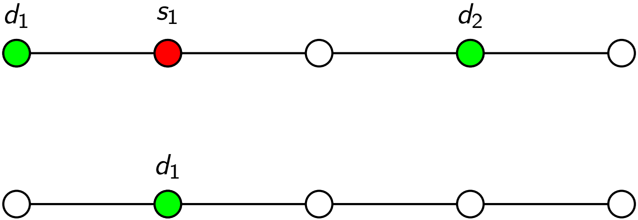
The game on P_5



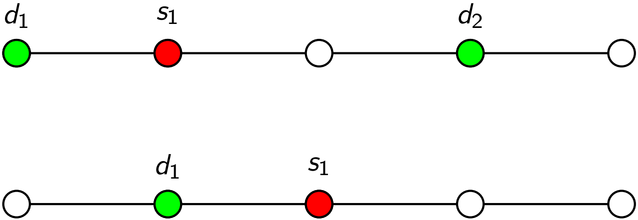
The game on P_5



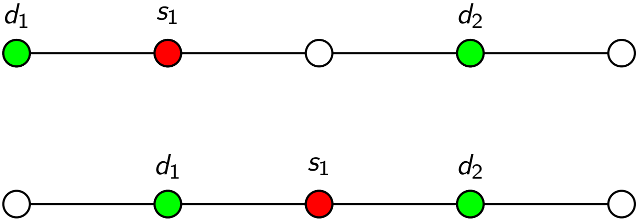
The game on P_5



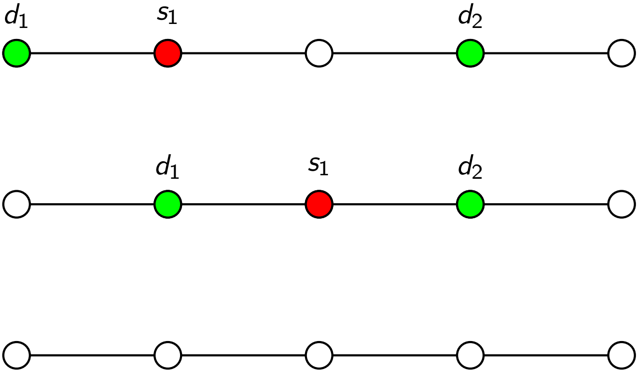
The game on P_5



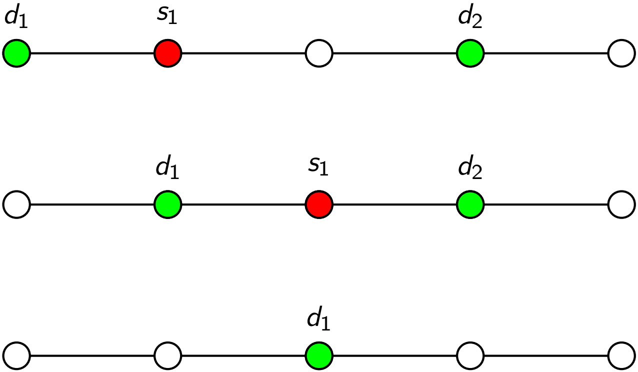
The game on P_5



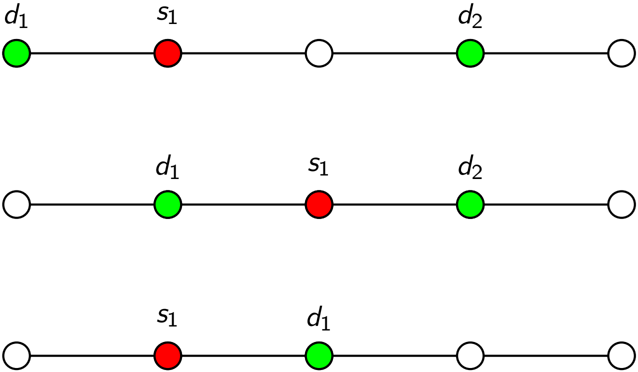
The game on P_5



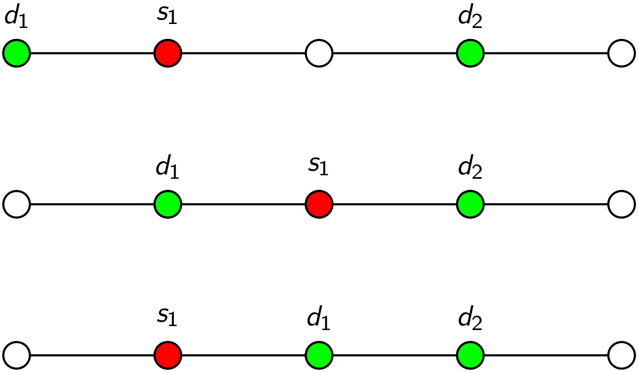
The game on P_5



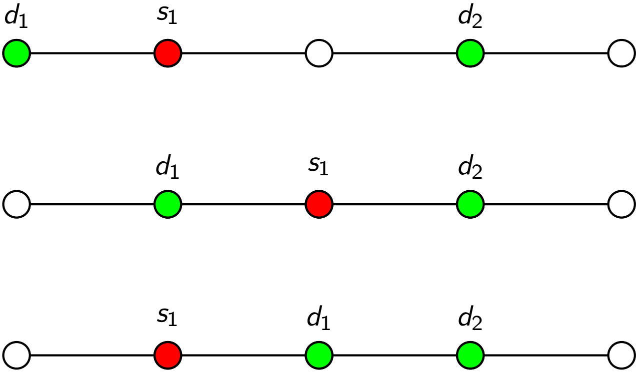
The game on P_5



The game on P_5



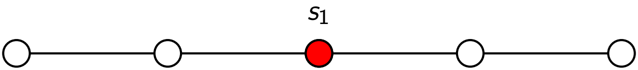
The game on P_5



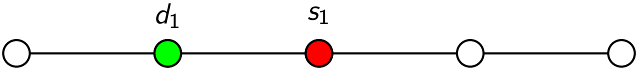
$$\gamma_g(P_5) = 3$$

The game on P_5 cont'd

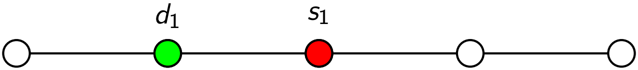
The game on P_5 cont'd



The game on P_5 cont'd

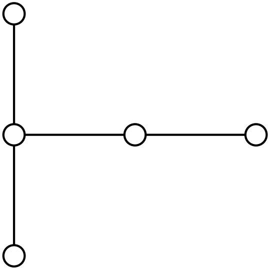


The game on P_5 cont'd

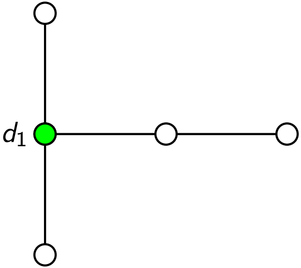


$$\gamma_g(P_5) = 3 = \gamma_g'(P_5)$$

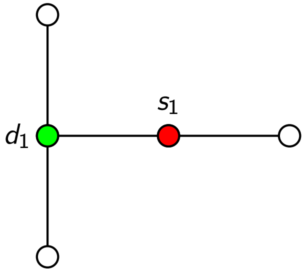
The game on a tree



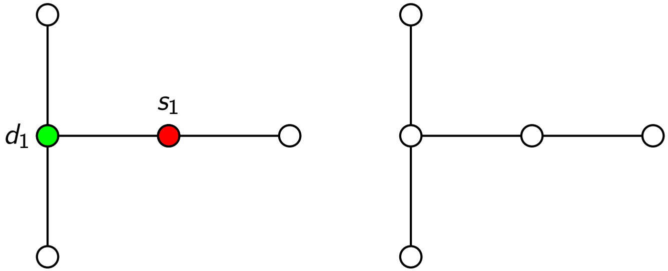
The game on a tree



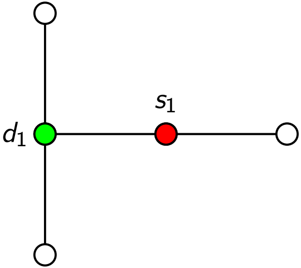
The game on a tree

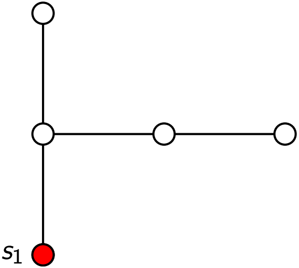


The game on a tree

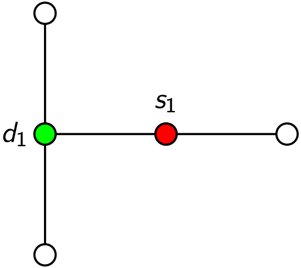


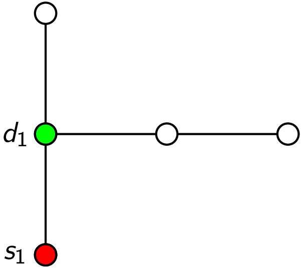
The game on a tree



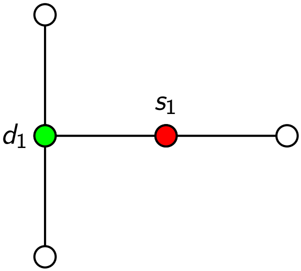


The game on a tree

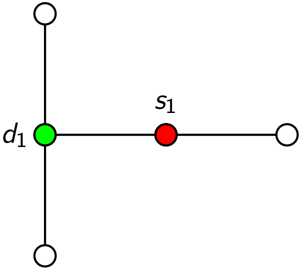




The game on a tree

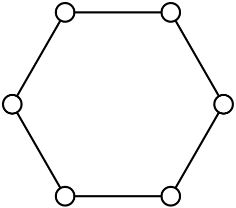


The game on a tree

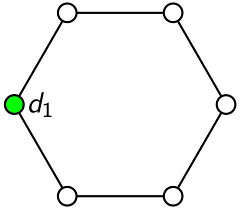


$$\gamma_g(T) = 2, \quad \gamma'_g(T) = 3$$

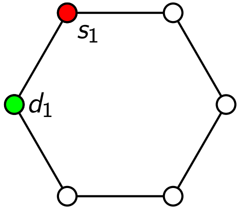
The game on C_6



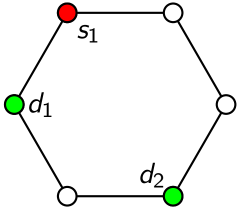
The game on C_6



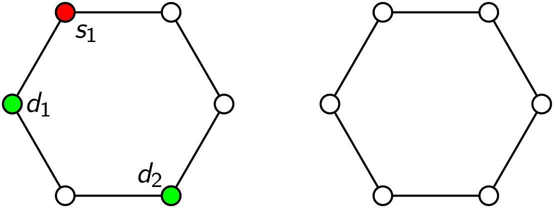
The game on C_6



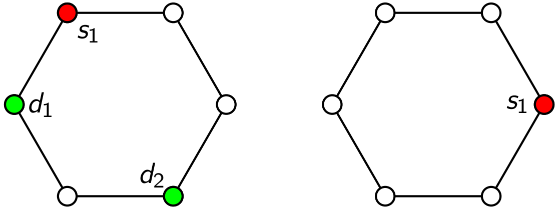
The game on C_6



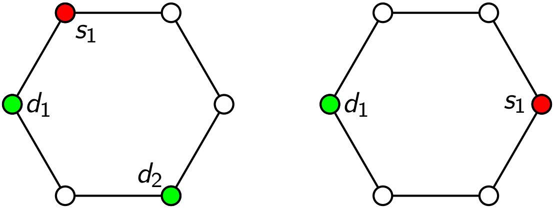
The game on C_6



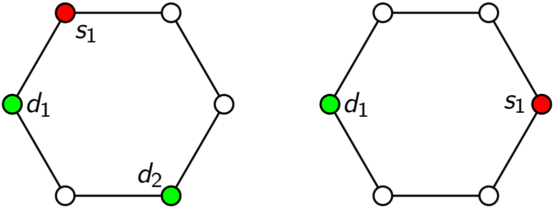
The game on C_6



The game on C_6



The game on C_6



$$\gamma_g(C_6) = 3, \quad \gamma'_g(C_6) = 2$$

Relations between invariants

γ_g versus γ

Suppose $\mathcal{D}$ and $\mathcal{S}$ play Game 1 on G.

γ_g versus γ

Suspose $\mathcal{D}$ and $\mathcal{S}$ play Game 1 on G.

- Let C denote the total set of vertices chosen by $\mathcal{D}$ and $\mathcal{S}$. Then C is a dominating set of G.

γ_g versus γ

Suppose $\mathcal{D}$ and $\mathcal{S}$ play Game 1 on G.

- Let C denote the total set of vertices chosen by $\mathcal{D}$ and $\mathcal{S}$. Then C is a dominating set of G.
- If $\mathcal{D}$ employs a strategy of selecting vertices from a minimum dominating set A of G, then Game 1 will have ended when $\mathcal{D}$ has exhausted the vertices from A.

γ_g versus γ

Suppose $\mathcal{D}$ and $\mathcal{S}$ play Game 1 on G.

- Let C denote the total set of vertices chosen by $\mathcal{D}$ and $\mathcal{S}$. Then C is a dominating set of G.
- If $\mathcal{D}$ employs a strategy of selecting vertices from a minimum dominating set A of G, then Game 1 will have ended when $\mathcal{D}$ has exhausted the vertices from A.

Theorem (Brešar, K., Rall, 2010)

If G is any graph, then $\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1$. Moreover, for any integer $k \geq 1$ and any $0 \leq r \leq k-1$, there exists a graph G with $\gamma(G) = k$ and $\gamma_g(G) = k + r$.

γ_g versus γ_g'

Theorem (Brešar, K., Rall, 2010; Kinnersley, West, Zamani, 2013?)

For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.

Theorem (Brešar, K., Rall, 2010; Kinnersley, West, Zamani, 2013?)

For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \le 1$.

- A partially dominated graph is a graph in which some vertices have already been dominated in some turns of the game already played.

γ_g versus γ_g'

Theorem (Brešar, K., Rall, 2010; Kinnersley, West, Zamani, 2013?)

For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.

- A partially dominated graph is a graph in which some vertices have already been dominated in some turns of the game already played.
- If X is a partially dominated graph, then $\gamma_g(X)$ ($\gamma_g'(X)$) is the number of turns remaining if $\mathcal{D}$ ($\mathcal{S}$) has the move.

Theorem (Brešar, K., Rall, 2010; Kinnersley, West, Zamani, 2013?)

For any graph G, $|\gamma_g(G) - \gamma_g'(G)| \leq 1$.

- A partially dominated graph is a graph in which some vertices have already been dominated in some turns of the game already played.
- If X is a partially dominated graph, then $\gamma_g(X)$ ($\gamma_g'(X)$) is the number of turns remaining if $\mathcal{D}$ ($\mathcal{S}$) has the move.

Lemma (Kinnersley, West, Zamani, 2013?)

(Continuation Principle) Let G be a graph and $A, B \subseteq V(G)$. Let G_A and G_B be partially dominated graphs in which the sets A and B have already been dominated, respectively. If $B \subseteq A$, then $\gamma_g(G_A) \leq \gamma_g(G_B)$ and $\gamma_g'(G_A) \leq \gamma_g'(G_B)$.

Proof of Continuation Principle

- $\mathcal{D}$ will play two games: Game A on G_A (real game) and Game B on G_B (imagined game).

Proof of Continuation Principle

- $\mathcal{D}$ will play two games: Game A on G_A (real game) and Game B on G_B (imagined game).
- $\mathcal{D}$ will keep the rule: the set of vertices dominated in Game A is a superset of vertices dominated in Game B.

Proof of Continuation Principle

- $\mathcal{D}$ will play two games: Game A on G_A (real game) and Game B on G_B (imagined game).
- $\mathcal{D}$ will keep the rule: the set of vertices dominated in Game A is a superset of vertices dominated in Game B.
- Suppose Game B is not yet finished. If there are no undominated vertices in Game A, then Game A has finished before Game B and we are done.

Proof of Continuation Principle

- $\mathcal{D}$ will play two games: Game A on G_A (real game) and Game B on G_B (imagined game).
- $\mathcal{D}$ will keep the rule: the set of vertices dominated in Game A is a superset of vertices dominated in Game B.
- Suppose Game B is not yet finished. If there are no undominated vertices in Game A, then Game A has finished before Game B and we are done.
- It is $\mathcal{D}$'s move: he selects an optimal move in game B. If it is legal in Game A, he plays it there as well, otherwise he plays any undominated vertex.

Proof of Continuation Principle cont'd

- It is $\mathcal{S}$'s move: she plays in Game A. By the rule, this move is legal in Game B and $\mathcal{D}$ can replicate it in Game B.

Proof of Continuation Principle cont'd

- It is $\mathcal{S}$'s move: she plays in Game A. By the rule, this move is legal in Game B and $\mathcal{D}$ can replicate it in Game B.
- By the rule, Game A finishes no later than Game B.

Proof of Continuation Principle cont'd

- It is $\mathcal{S}$'s move: she plays in Game A. By the rule, this move is legal in Game B and $\mathcal{D}$ can replicate it in Game B.
- By the rule, Game A finishes no later than Game B.
- $\mathcal{D}$ played optimally on Game B. Hence:
  - If $\mathcal{D}$ played first in Game B, the number of moves taken on Game B was at most $\gamma_g(G_B)$ (indeed, if $\mathcal{S}$ did not play optimally, it might be strictly less);
  - If $\mathcal{S}$ played first in Game B, the number of moves taken on Game B was at most $\gamma_g'(G_B)$.

Proof of Continuation Principle cont'd

- It is $\mathcal{S}$'s move: she plays in Game A. By the rule, this move is legal in Game B and $\mathcal{D}$ can replicate it in Game B.
- By the rule, Game A finishes no later than Game B.
- $\mathcal{D}$ played optimally on Game B. Hence:
  - If $\mathcal{D}$ played first in Game B, the number of moves taken on Game B was at most $\gamma_g(G_B)$ (indeed, if $\mathcal{S}$ did not play optimally, it might be strictly less);
  - If $\mathcal{S}$ played first in Game B, the number of moves taken on Game B was at most $\gamma_g'(G_B)$.
- Hence
  - If $\mathcal{D}$ played first in Game B, then $\gamma_g(G_A) \leq \gamma_g(G_B)$;
  - If $\mathcal{S}$ played first in Game B, then $\gamma_g'(G_A) \leq \gamma_g'(G_B)$.

Proof of the theorem

- Consider Game 1 and let v be the first move of $\mathcal{D}$.

Proof of the theorem

- Consider Game 1 and let v be the first move of $\mathcal{D}$.
- Let G' be the resulting partially dominated graph.

Proof of the theorem

- Consider Game 1 and let v be the first move of $\mathcal{D}$.
- Let G' be the resulting partially dominated graph.
- $\gamma_g(G) \leq \gamma_g'(G') + 1$.

Proof of the theorem

- Consider Game 1 and let v be the first move of $\mathcal{D}$.
- Let G' be the resulting partially dominated graph.
- $\gamma_g(G) \leq \gamma_g'(G') + 1$.
- By Continuation Principle, $\gamma_g'(G') \leq \gamma_g'(G)$.

Proof of the theorem

- Consider Game 1 and let v be the first move of $\mathcal{D}$.
- Let G' be the resulting partially dominated graph.
- $\gamma_g(G) \leq \gamma_g'(G') + 1$.
- By Continuation Principle, $\gamma_g'(G') \leq \gamma_g'(G)$.
- Hence $\gamma_g(G) \leq \gamma_g'(G') + 1 \leq \gamma_g'(G) + 1$.

By a parallel argument, $\gamma_g'(G) \leq \gamma_g(G) + 1$.

Realizable pairs

A pair (r, s) of integers is realizable if there exists a graph G such that $\gamma_g(G) = r$ and $\gamma_g'(G) = s$.

Realizable pairs

A pair (r, s) of integers is realizable if there exists a graph G such that $\gamma_g(G) = r$ and $\gamma_g'(G) = s$. By the theorem, only possible realizable pairs are: (r, r), $(r, r+1)$, $(r, r-1)$.

Realizable pairs

A pair (r, s) of integers is realizable if there exists a graph G such that $\gamma_g(G) = r$ and $\gamma_g'(G) = s$. By the theorem, only possible realizable pairs are: (r, r), $(r, r+1)$, $(r, r-1)$.

Theorem (Košmrlj, 2014)

Pairs $(r, r), r \geq 2$, $(r, r+1), r \geq 1$, and $(2k, 2k-1), k \geq 2$, are realizable by 2-connected graphs. Pairs $(2k+1, 2k), k \geq 2$ are realizable by connected graphs.

Realizable pairs

A pair (r, s) of integers is realizable if there exists a graph G such that $\gamma_g(G) = r$ and $\gamma'_g(G) = s$. By the theorem, only possible realizable pairs are: (r, r), $(r, r+1)$, $(r, r-1)$.

Theorem (Košmrlj, 2014)

Pairs $(r, r), r \geq 2$, $(r, r+1), r \geq 1$, and $(2k, 2k-1), k \geq 2$, are realizable by 2-connected graphs. Pairs $(2k+1, 2k), k \geq 2$ are realizable by connected graphs.

Theorem (Kinnerley, 2014?)

No pair of the form $(r, r-1)$ can be realized by a tree.

Game on spanning trees

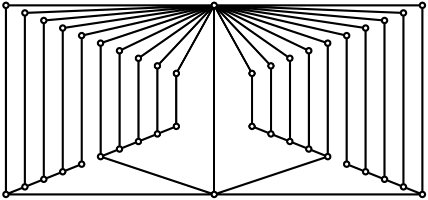
Theorem (Brešar, K., Rall, 2013)

For any integer $\ell \geq 1$, there exists a graph G and its spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$.

Theorem (Brešar, K., Rall, 2013)

For any integer $\ell \geq 1$, *there exists a graph* G *and its spanning tree* T *such that* $\gamma_g(G) - \gamma_g(T) \geq \ell$.

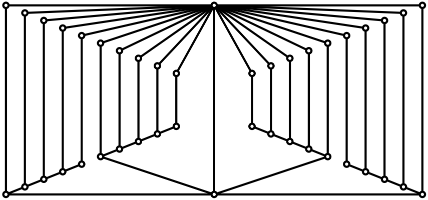
G_4:



Theorem (Brešar, K., Rall, 2013)

For any integer $\ell \geq 1$, there exists a graph G and its spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$.

G_4:

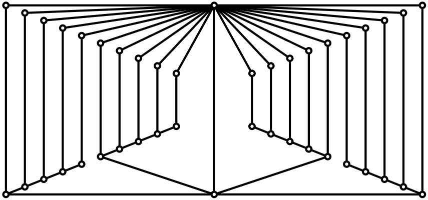


- T_k: in G_k remove all but the middle vertical edges.

Theorem (Brešar, K., Rall, 2013)

For any integer $\ell \geq 1$, there exists a graph G and its spanning tree T such that $\gamma_g(G) - \gamma_g(T) \geq \ell$.

G_4:



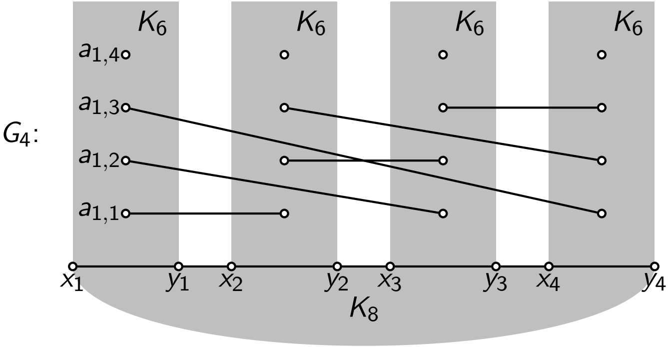
- T_k: in G_k remove all but the middle vertical edges.
- $\gamma_g(G_k) \geq \frac{5}{2}k - 1 \quad \text{and} \quad \gamma_g(T_k) \leq 2k + 3$.

Game on spanning subgraphs

Theorem (Brešar, K., Rall, 2013)

For any $m \geq 3$ there exists a 3-connected graph G_m and its 2-connected spanning subgraph H_m such that $\gamma_g(G_m) \geq 2m - 2$ and $\gamma_g(H_m) = m$.

Game on spanning subgraphs cont'd



- H_m is obtained from G_m by removing all the edges $a_{i,j}a_{j+1,i}$.

Open problems

3/5-conjectures

Conjecture (Kinnersley, West, Zamani, 2013?)

For an n-vertex forest T without isolated vertices,

$$\gamma_g(T) \leq \frac{3n}{5} \quad \text{and} \quad \gamma_g'(T) \leq \frac{3n+2}{5}\,.$$

3/5-conjectures

Conjecture (Kinnersley, West, Zamani, 2013?)

For an n-vertex forest T without isolated vertices,

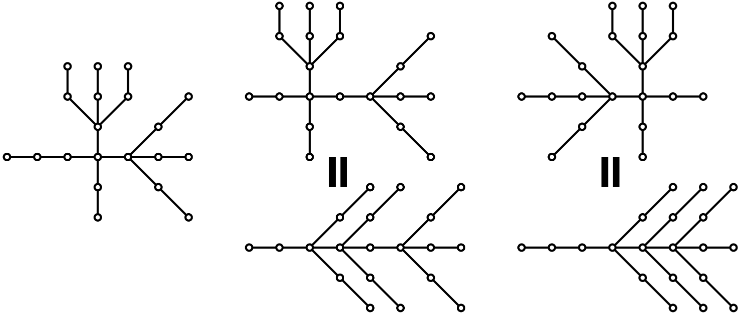
$$\gamma_g(T) \leq \frac{3n}{5} \quad \text{and} \quad \gamma_g'(T) \leq \frac{3n+2}{5}.$$

Conjecture (Kinnersley, West, Zamani, 2013?)

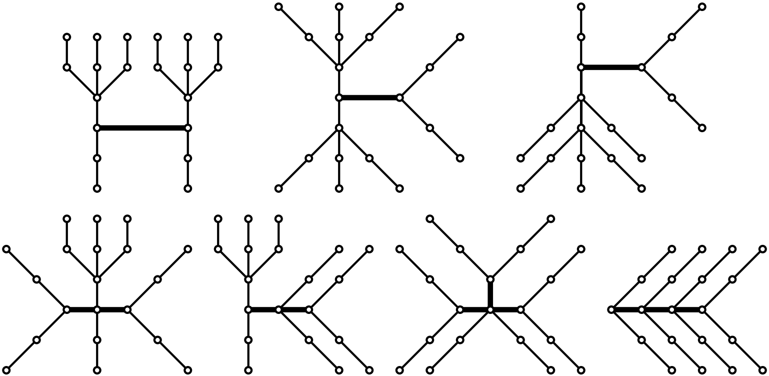
For an n-vertex connected graph G,

$$\gamma_g(G) \leq \frac{3n}{5} \quad \text{and} \quad \gamma_g'(G) \leq \frac{3n+2}{5}.$$

3/5-trees on 20 vertices



3/5-trees on 20 vertices cont'd



3/5-conjectures con't

Theorem (Bujtás, 2014?)

The 3/5-conjecture holds true for forests in which no two leaves are at distance 4.

Computational complexity

Problem

What is the computational complexity of the game domination number?

Computational complexity

Problem

What is the computational complexity of the game domination number?

Problem

What is the computational complexity of the game domination number on trees?

Computational complexity

Problem

What is the computational complexity of the game domination number?

Problem

What is the computational complexity of the game domination number on trees?

Problem

Can we say **anything** *about the computational complexity of the domination game?*

Game Over!