Cubic vertices in minimal bricks

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Joint work with Maya Stein

Bricks

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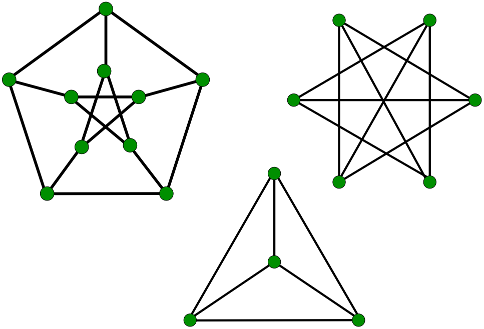

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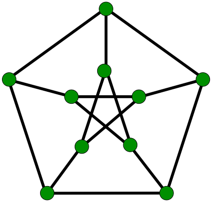

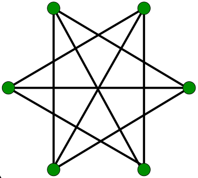

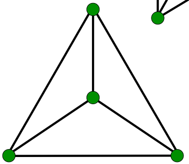

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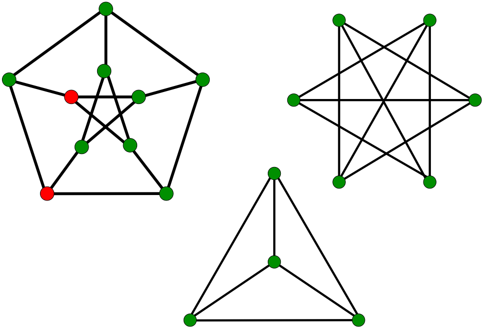

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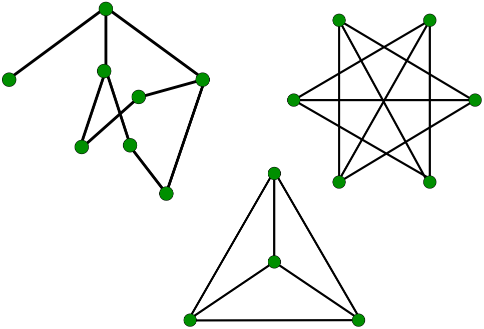

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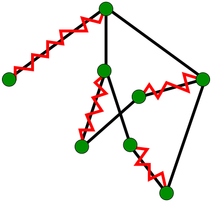

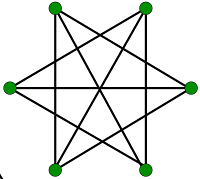

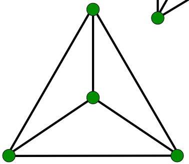

Tight Cut Decomposition - Motivation

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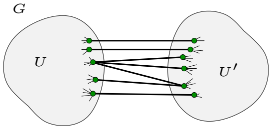

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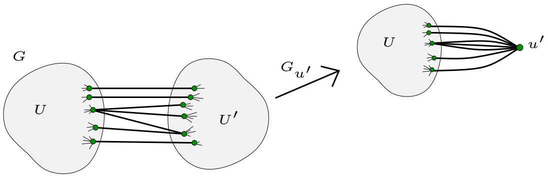

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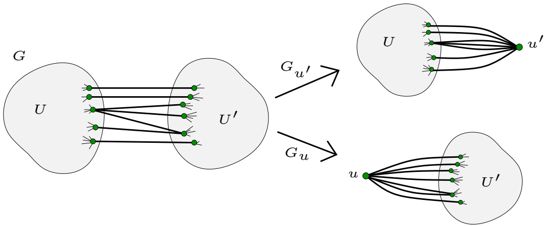

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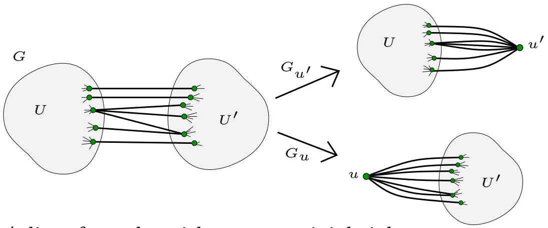

- Output: A list of graphs without non-trivial tight cuts.

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Theorem [Lovász 1986]

The list of bricks and braces is unique.

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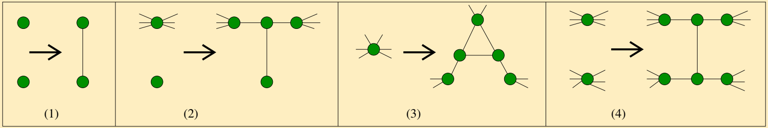
- Pfaffian bricks — Norine (Ph.D. Thesis) 2005

Theorem [Carvalho, Lucchesi & Murty 2004]

Every brick can be obtained from one of the basic bricks by a sequence of applications of the following four operations (expansions):



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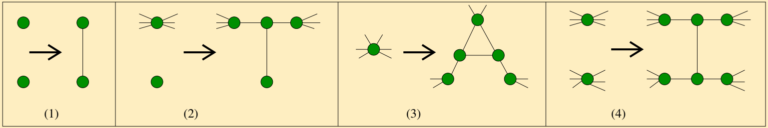

Corollary (Lovász's Conjecture)

Every minimal brick has a vertex of degree 3.

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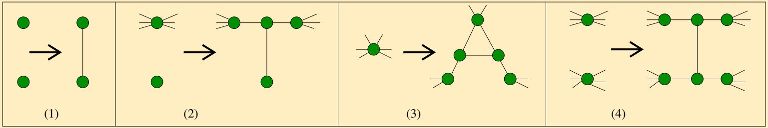

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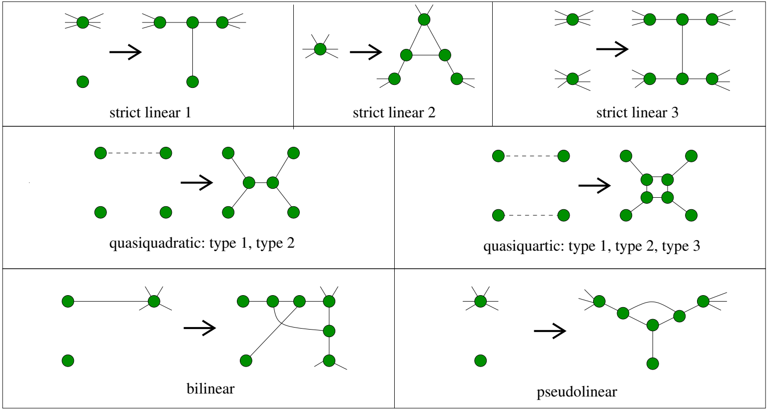
- More about brick generation — Norine & Thomas

Theorem [Norine & Thomas 2005]

Every minimal brick other than the Petersen graph can be obtained from K_4 or $\bar{C}_6$ by a sequence of applications of strict extensions.



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Every minimal brick G has at least $\frac{|V(G)|}{9}$ vertices of degree 4.

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Theorem [J. & Stein 2013]

Every minimal brick G has at least $\frac{1}{52}\sqrt{|V(G)|}$ vertices of degree 3.

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Generous, neutral and selfish operations

$i \in \{1, \dots, k\}, \quad p = p(i) = |G_i|_3 - |G_{i-1}|_3$

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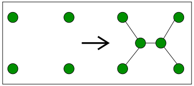

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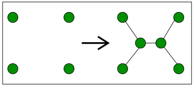

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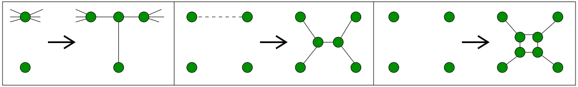

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Average degree

$i \in \{1, \dots, k\}$ we define

$$n(i) := |V(G_i)| - |V(G_{i-1}| \quad e(i) := |E(G_i)| - |E(G_{i-1})| \quad d(i) := 2\frac{e(i)}{n(i)}$$

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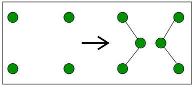

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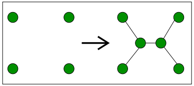

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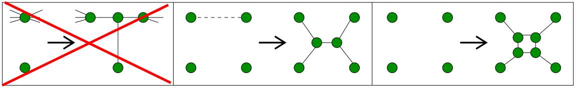

Proof's Ideas

- $G_0 \overset{\psi_1}{\longmapsto} G_1 \overset{\psi_2}{\longmapsto} \dots \overset{\psi_k}{\longmapsto} G_k :=$ nice min-brick sequence
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Lemma

There exists a partition I_s^a, I_s^b of I_s such that

(a) for each $i \in I_s^a$ there is a vertex v_i that has degree 3 in G and the v_i's are distinct for distinct $i \in I_s^a$, and

(b) there is $\tilde{I}_s^b \subseteq \{1, \dots, k\}$ such that $I_s^b \subseteq \tilde{I}_s^b$ and

$$\sum_{j \in \tilde{I}_s^b} (|G_j|_3 - |G_{j-1}|_3) \geq \frac{1}{4}|I_s^b|.$$

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  (ii) subcase (i) does not happen.

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(ii) $\sim$ Case 1; taking bad subsequences instead of isolated operations.

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(i) A bad subsequence of lenght at least $\frac{27}{52}\sqrt{k}$ gives at least $\frac{27}{52}\sqrt{k} - 2|I_s| \geq \frac{1}{26}\sqrt{k}$ vertices of degree 3 in G.

(ii) $\sim$ Case 1; taking bad subsequences instead of isolated operations.

Case 2.2: $|\{1, \ldots, k\} - I_s - I_n| \geq \frac{7}{13}\sqrt{k}$

Case 2: $|I_s| < \frac{1}{2}\sqrt{k}$

- $I_n \subset \{1, 2, \ldots, k\}$, with $j \in I_n$ for ψ_j neutral and $d(j) = 4$

Case 2.1: $|I_n| \geq k - \frac{27}{26}\sqrt{k}$

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(i) A bad subsequence of lenght at least $\frac{27}{52}\sqrt{k}$ gives at least $\frac{27}{52}\sqrt{k} - 2|I_s| \geq \frac{1}{26}\sqrt{k}$ vertices of degree 3 in G.

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- $i \in |\{1, \ldots, k\} - I_s - I_n|$, then $d(i) \leq 3.5$

Gracias :-)