Splittability of Permutation Classes

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Permutation Containment

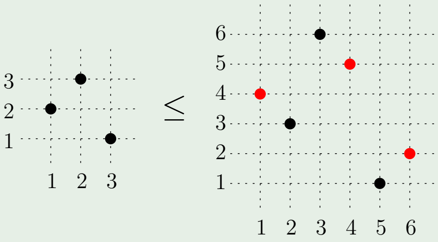
Definition

A permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ contains a permutation $\sigma = \sigma_1 \cdots \sigma_m$, denoted $\sigma \leq \pi$, if π has a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}$ such that for all $1 \leq j < k \leq m$

$$\sigma_j < \sigma_k \iff \pi_{i_j} < \pi_{i_k}.$$

Example

$$231 \leq 436512$$



Permutation Classes

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- A permutation class is a set of permutations $\mathcal{C}$ such that for every $\pi \in \mathcal{C}$ and $\sigma \leq \pi$, we have $\sigma \in \mathcal{C}$.
- Av(σ) is the class of all permutations avoiding σ.
- A principal permutation class is the class of the form Av(σ) for some σ.

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Merging Permutations

Definition

Permutation π is a merge of permutations σ and τ if the symbols of π can be colored red and blue, so that the red symbols are a copy of σ and the blue ones of τ.

Example

3175624 is a merge of 231 and 1342.

Definition

For two sets P and Q of permutations, let $P \odot Q$ be the set of permutations obtained by merging a $\sigma \in P$ with a $\tau \in Q$.

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A permutation class $\mathcal{C}$ is splittable if it has two proper subclasses $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{C} \subseteq \mathcal{A} \odot \mathcal{B}$. (We then say that $\mathcal{C}$ is split into $\mathcal{A}$ and $\mathcal{B}$.)

Example

Every permutation avoiding 123 is a union of two decreasing subsequences, i.e., $\text{Av}(123) \subseteq \text{Av}(12) \odot \text{Av}(12)$. Therefore, $\text{Av}(123)$ is splittable.

Example

$\text{Av}(12)$ is not splittable, since all its proper subclasses are finite.

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Simple Lemma

The following are equivalent:

- $\mathcal{C}$ *is splittable.*
- *For some* k, $\mathcal{C}$ *has* k *proper subclasses* $\mathcal{A}_1, \ldots, \mathcal{A}_k$ *such that* $\mathcal{C} \subseteq \mathcal{A}_1 \odot \cdots \odot \mathcal{A}_k$.
- *There are two permutations* $\sigma, \pi \in \mathcal{C}$ *such that* $\mathcal{C} \subseteq Av(\sigma) \odot Av(\pi)$.

Corollary

$\mathcal{C}$ *is unsplittable* $\iff$ *for every* $\sigma, \pi \in \mathcal{C}$ *there is* $\rho \in \mathcal{C}$ *such that any red-blue coloring of* ρ *has a red copy of* σ *or a blue copy of* π.

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The Plan

Outline of the talk:

- Part I: Which (principal) classes are splittable?
- Part II: Relation to other Ramsey-type properties
- (Optional part III: Relation to χ-boundedness of circle graphs)

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Direct Sums

Definition

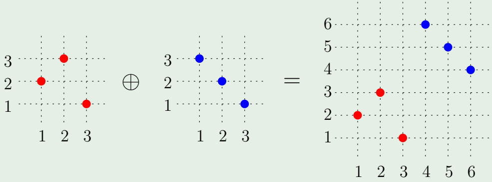
Given two permutations $\pi = \pi_1, \ldots, \pi_k$ and $\sigma = \sigma_1, \ldots, \sigma_m$, define the direct sum $\pi \oplus \sigma$ as

$$\pi \oplus \sigma = \pi_1, \ldots, \pi_k, \sigma_1 + k, \ldots, \sigma_m + k.$$

A permutation is $\oplus$-decomposable if it is a direct sum of two nonempty permutations.

Example

$231 \oplus 321 = 231654$



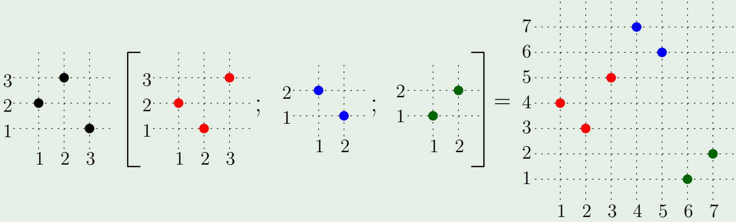
Inflations

Definition

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation and $\sigma_1,\ldots,\sigma_n$ a sequence of n permutations. The inflation of π by $\sigma_1,\ldots,\sigma_n$, denoted by $\pi[\sigma_1;\ldots;\sigma_n]$ is the permutation obtained by replacing each π_i by a copy $\overline{\sigma_i}$ of σ_i, so that if $\pi_i < \pi_j$, then all elements of $\overline{\sigma_i}$ are smaller than those of $\overline{\sigma_j}$.

Example

231[213; 21; 12]= 4357612



Inflations and Simple Permutations

Definition

A class of permutations $\mathcal{C}$ is closed under inflations if $\pi[\sigma_1; \dots ; \sigma_n]$ belongs to $\mathcal{C}$ whenever all the permutations $\pi, \sigma_1, \dots, \sigma_n$ belong to $\mathcal{C}$.

Definition

A permutation is simple if it cannot be obtained from smaller permutations by inflation.

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Splittability Criteria

These classes are unsplittable:

- Av(σ) for a simple permutation σ
- more generally, any $\mathcal{C}$ closed under inflations
- Av(213)

These classes are splittable:

- Av(σ) for a $\oplus$-decomposable σ of size at least 4

Open problem

If σ is neither simple nor $\oplus$-decomposable, is Av(σ) splittable?

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Theorem

If σ is $\oplus$-decomposable of size at least 4, then Av(σ) is splittable.

- Proof of the theorem is difficult, splittings have a complicated structure.
- Some special cases are easier: for all α, β, γ, we have Av$(\alpha \oplus \beta \oplus \gamma) \subseteq$ Av$(\alpha \oplus \beta) \odot$ Av$(\beta \oplus \gamma)$.
- Corollary: Av$(\alpha \oplus \beta) \subseteq$ Av$(\alpha \oplus 1) \odot$ Av$(1 \oplus \beta)$.
- The hard part: show that Av$(\alpha \oplus 1)$ is splittable for α of size ≥ 3.

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Ramsey-type properties

Definition

A permutation class $\mathcal{C}$ is atomic if for every $\sigma, \pi \in \mathcal{C}$, there is a $\rho \in \mathcal{C}$ which contains both σ and π.

Fact

- $\mathcal{C}$ *is atomic* $\iff$ $\mathcal{C}$ *cannot be expressed as a union of two of its proper subclasses.*
- *A non-atomic class is trivially splittable.*
- *All principal classes are atomic.*

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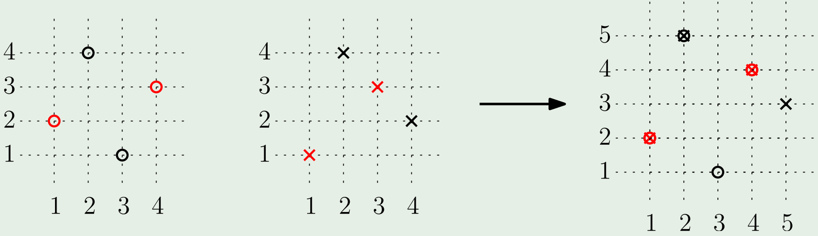
Amalgamations

Definition

Let σ_1 and σ_2 be two permutations, each having a prescribed occurrence of a permutation π. An amalgamation of σ_1 and σ_2 is a permutation obtained from σ_1 and σ_2 by identifying the two prescribed occurrences of π (and possibly identifying some more elements as well).

Example (with $\pi = 12$)

$\sigma_1 = 2413$ and $\sigma_2 = 1432$ admit an amalgamation 25143



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- amalgamable if it is π-amalgamable for every $\pi \in \mathcal{C}$.
- k-amalgamable for $k \in \mathbb{N}$, if it is π-amalgamable for every $\pi \in \mathcal{C}$ of size at most k.

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Amalgamable classes (continued)

Theorem (Cameron, 2002)

There are five nontrivial amalgamable permutation classes: $Av(12)$, $Av(21)$, $Av(231, 312)$, $Av(132, 213)$, *and the class of all permutations.*

Fact

- *All 3-amalgamable permutation classes are amalgamable.*
- *Each unsplittable class is 1-amalgamable.*

Open problems

- Which classes are 2-amalgamable? Are there infinitely many?
- Is there a splittable 1-amalgamable class?

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Ramseyness

Notation: $\binom{\sigma}{\pi}$ is the set of all occurrences of π in σ.

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- A class $\mathcal{C}$ is π-Ramsey if for $\sigma \in \mathcal{C}$ there is a $\rho \in \mathcal{C}$ such that any 2-coloring of $\binom{\rho}{\pi}$ has a monochromatic copy of σ.
- A class $\mathcal{C}$ is Ramsey if it is π-Ramsey for every $\pi \in \mathcal{C}$.
- A class $\mathcal{C}$ is k-Ramsey if it is π-Ramsey for every $\pi \in \mathcal{C}$ of size at most k.

Fact

Suppose $\mathcal{C}$ is an atomic permutation class.

- *If $\mathcal{C}$ is π-Ramsey, then it is π-amalgamable [Nešetřil].*
- *$\mathcal{C}$ is 1-Ramsey $\iff$ $\mathcal{C}$ is unsplittable.*

Ramseyness

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Ramseyness (continued)

Theorem (Böttcher & Foniok, 2013)

All amalgamable permutation classes are Ramsey.

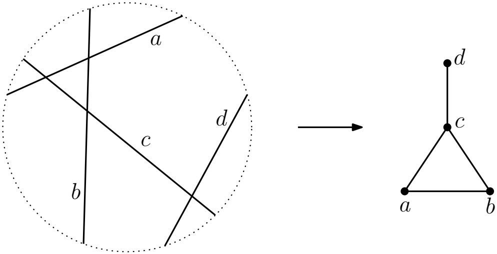
Open problem

Is there a permutation class that is k-amalgamable but not k-Ramsey? (The answer is NO if $k \geq 3$.)

Circle Graphs

Definition

A graph G is a circle graph if it is the intersection graph of a collection of chords on a circle.



Colorings of Circle Graphs

Theorem (Gyárfás, 1985)

For every k there is a c such that every circle graph with no clique of size k can be colored by c colors.

Let $c(k)$ be the smallest c with the above property.

Theorem

Let λ_k be the permutation $k(k-1)\cdots 1(k+1)$. The class $Av(\lambda_k)$ can be split into $c(k)$ copies of $Av(213)$, and this is optimal.

- $c(k) \leq 2^{O(k)}$ [Gyárfás; Kostochka & Kratochvíl; Černý]
- $c(k) \geq \Omega(k \log k)$ [Kostochka]
- $c(3) = 5$ [Ageev; Kostochka] and $c(4) \leq 30$ [Nenashev]

Open problem

What is $c(k)$?

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The End

Thank you for your attention!