Polynomial graph invariants from graph homomorphisms

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Overview

Graph polynomials with a name for themselves...

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... polynomials determined by counting H_k-colourings of a graph for a sequence of (multi)graphs $(H_k : k = 1, 2, \dots)$
e.g. for $k \in \mathbb{N}$, $P(G;k)$ counts K_k-colourings

Definition

Graphs G, H.
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$$\begin{aligned}\hom(G,H) &= \#\{\text{homomorphisms from } G \text{ to } H\}\\ &= \#\{H\text{-colourings of } G\}\end{aligned}$$

when H simple ($a_{s,t} \in \{0,1\}$) or multigraph ($a_{s,t} \in \mathbb{N}$)

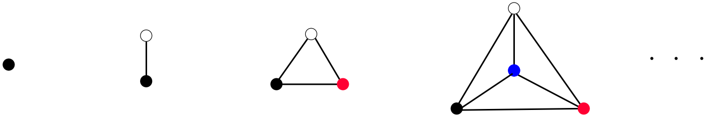
The main question

For sequence $(H_{k,\ell,\dots})$, when is, for all graphs G,

$$\hom(G, H_{k,\ell,\dots}) = p(G; k, \ell, \dots)$$

for polynomial $p(G)$?

Examples



$$(K_k)$$

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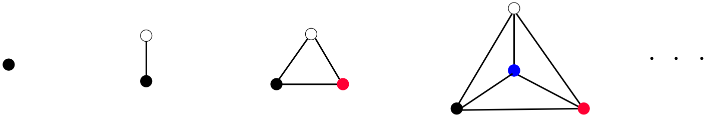

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$$\hom(G, K_k) = P(G; k)$$

chromatic polynomial

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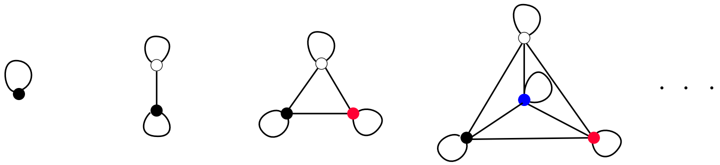
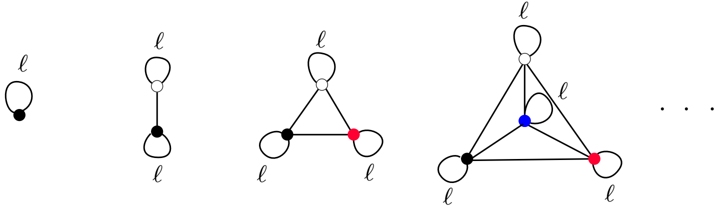


$$(K_k^1)$$

$$\hom(G, K_k^1) = k^{|V(G)|}$$

Examples



$$(K_k^\ell)$$

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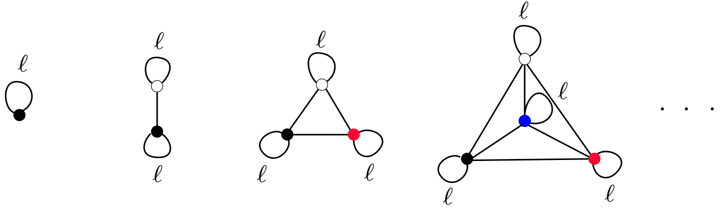

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$$= k^{c(G)}(\ell-1)^{r(G)}T(G; \tfrac{\ell-1+k}{\ell-1}, \ell)$$

It has something to do with automorphisms...

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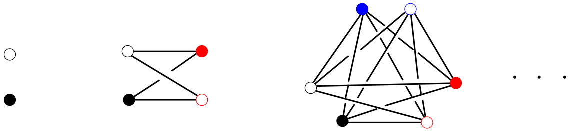

$$(\overline{kK_2}) = (K_{2,\ldots,2})$$

$$\mathrm{Aut}(K_{2,\ldots,2}) \cong \mathrm{Sym}_k[\mathrm{Sym}_2]$$

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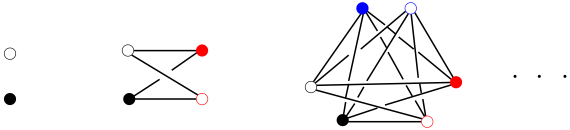

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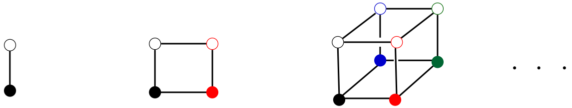
$$\hom(G, K_{2,\ldots,,2}) = 2^{|V(G)|} P(G; k)$$

... but what precisely?



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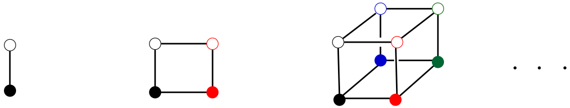

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Proposition (Garijo, G., Nešetřil, 2013+)

$\hom(G, Q_k) = p(G; k, 2^k)$ *for bivariate polynomial* $p(G)$

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- (C_k), (P_k) polynomial in k
- (Q_k) not polynomial in k (but in k and 2^k)

Subgraph criterion for strongly polynomial

$$\hom(G, H_k) = \sum_{\substack{S \subseteq H_k \\ |V(S)| \leq |V(G)|}} \mathrm{sur}_{\mathsf{V,E}}(G, S)$$

$$= \sum_{S/\cong} \mathrm{sur}_{\mathsf{V,E}}(G, S)\ \#\{\text{copies of } S \text{ in } H_k\}$$

Assuming G connected, homomorphic image S also connected

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Proposition (De la Harpe & Jaeger, 1995)

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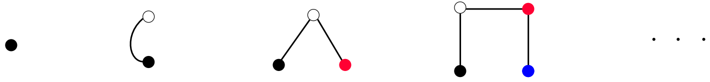
Polynomial but not strongly polynomial



$$(P_k)$$

$$\hom(G, P_k) = \sum_{1 \leq j \leq \min\{|V(G)|, k\}} \mathrm{sur}_{\mathrm{V}}(G, P_j)\,(k-j+1)$$

$\hom(P_4, P_2) = 2$, and $\hom(P_4, P_k) = 8k - 16$ for $k \geq 3$

Proposition (de la Harpe & Jaeger, 1995; Garijo, G., Nešetřil, 2013+)

If (H_k) strongly polynomial, H_k simple, then

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If (H_k) strongly polynomial, at most one loop each vertex of H_k, then

- (H_k^0) *(remove all loops)*
- (H_k^1) *(add loops to make 1 loop each vertex)*

strongly polynomial.
More generally, (H_k^ℓ) strongly polynomial in k, ℓ.

Proposition

If (F_j), (H_k) *strongly polynomial, then*

- $(F_j \cup H_k)$
- $(F_j + H_k)$

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$$\hom(G, K_{k-j}^1 + K_j^{\ell}) = \xi(G; k, \ell-1, -j(\ell-1))$$

Three-term recurrence: for $uv \in E(G)$,

$$\xi(G) = a\xi(G/uv) + b\xi(G\backslash uv) + c\xi(G-u-v)$$

Definition

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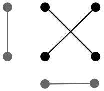
Example

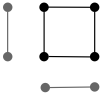
- $K_r[\{\overline{K_{k_1}}, \ldots, \overline{K_{k_r}}\}] \cong K_{k_1,\ldots,k_r}$ (*complete r-partite graph*)
- $F_{v;k_v} = F_k$ all $v \in V(H)$ gives lexicographic product $H[F_k]$

Graph products: direct, cartesian, lexicographic

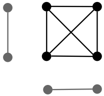
Graphs H, H', $\quad u, v \in V(H)$, $u', v' \in V(H')$



$H \times H'$ $\qquad$ $u \sim u'$ and $v \sim v'$



$H \,\square\, H'$ $\qquad$ $u = v$ and $u' \sim v'$,
or $u \sim v$ and $u' = v'$



$H[H']$ $\qquad$ $u \sim v$,
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Proposition (Garijo, G., Nešetřil, 2013+)

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- $(F_j \times H_k)$
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strongly polynomial in j, k.

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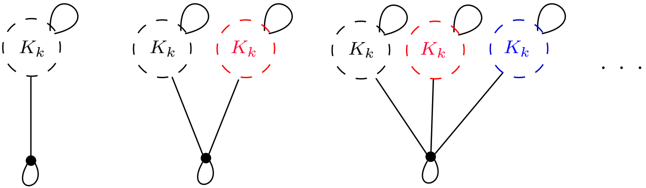
If (F_j), (H_k) strongly polynomial, is then $(F_j \square H_k)$ also?

A new type of strongly polynomial sequence



$$H_{j,k} = K_{1,j}[\{K_1^1\} \cup \{K_k^1 \text{ on leaves}\}]$$

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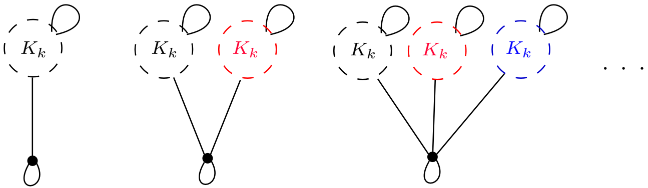

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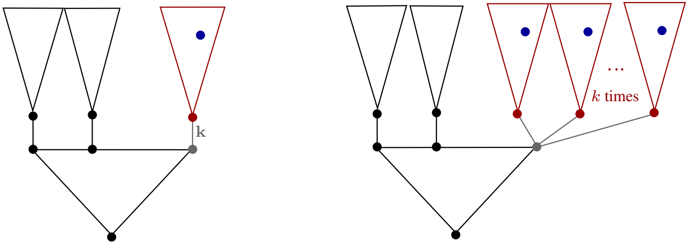
Tittmann–Averbouch–Godlin polynomial

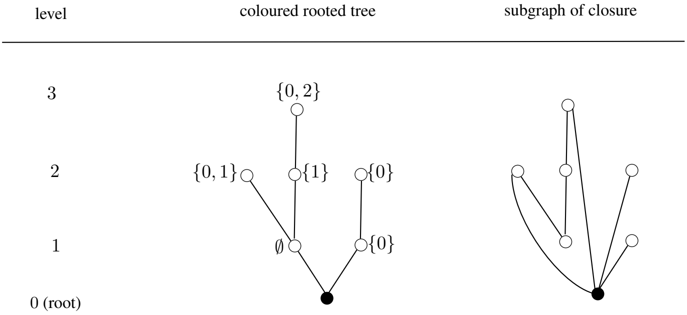
(includes independence polynomial, satisfies three-term recurrence)

Branching coloured rooted trees



"k-branching" at edge of coloured rooted tree

Colours encoding subgraph of closure of rooted tree



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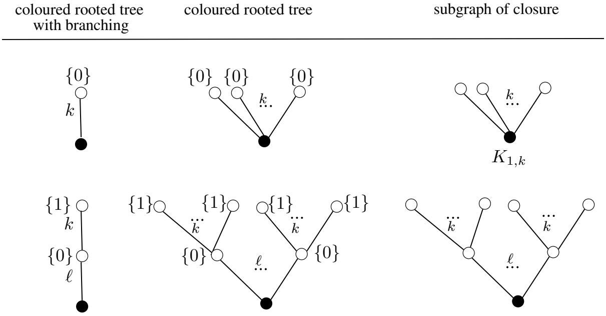

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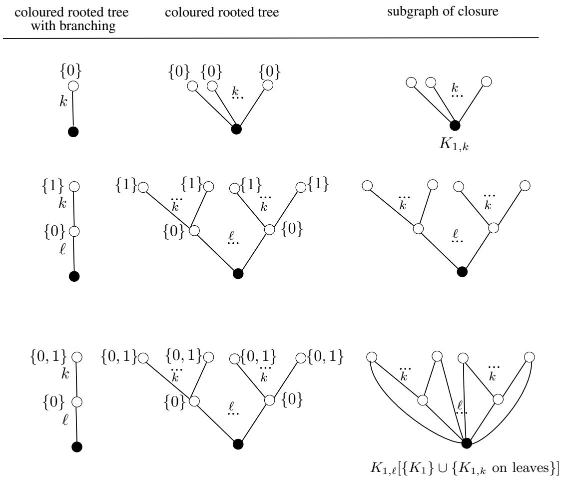
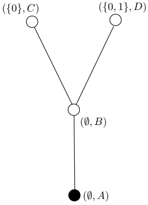
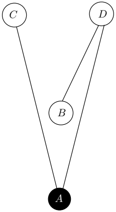
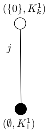
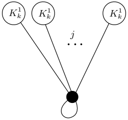


(2) Colours encoding subgraph along with ornaments



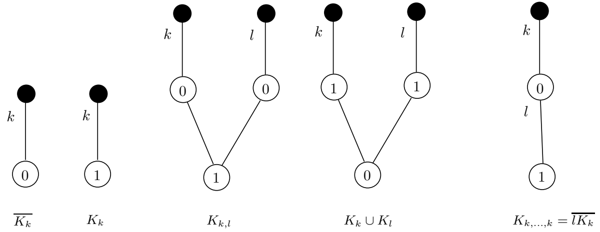






(Tittmann–Averbouch–Makowsky polynomial)

(3) Colours encoding cographs by cotrees



leaves = vertices of cograph

0 = disjoint union, 1 = join

Theorem (Garijo, G., Nešetřil, 2013+)

- *Coloured rooted tree* T *representing graph* H
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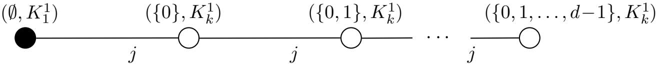
(3) cotree T encoding of cograph H,
colour non-leaf of T from $\{\cup, +\}$, leaves of $T = V(H)$

coloured rooted tree encoding graph $H_{j,k}$



(Closure of perfect j-ary tree)

$$\hom(G, H_{j,k}) = \sum_{\emptyset \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_d \subseteq V} j^{|W_d|} k^{\sum_{1 \le \ell \le d} c(G[W_\ell])}$$

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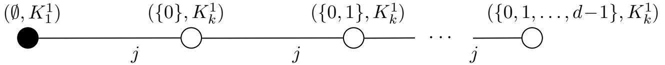

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Question

This bivariate polynomial generalizes the Tittmann– Averbouch–Makowsky polynomial.
Properties? Evaluations?

Definition

Generalized Johnson graph $J_{k,\ell,D}$, $D \subseteq \{0, 1, \ldots, \ell\}$
vertices $\binom{[k]}{\ell}$, edge uv when $|u \cap v| \in D$

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Question

Can generalized Johnson graphs be generated from simpler sequences by branching coloured rooted trees?

Some further questions

- Is there a characterization of strongly polynomial sequences (H_k) by the sequence of automorphism groups $(\mathrm{Aut}(H_k))$?

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- Can (H_k) be verified to be strongly polynomial by testing $\hom(G, H_k)$ for G only in a restricted class of graphs? (yes, for connected graphs – but for a smaller class?)
- Which graph polynomials defined by strongly polynomial sequences of graphs satisfy a reduction formula (size-decreasing recurrence) like the chromatic polynomial?

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