Restricted degree sequences

Péter L. Erdős

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences

Midsummer Combinatorial Workshop XIX,
Prague, July 27 – August 2, 2013

Social and biological networks

- exponential growth in network theory in last 15 years

Social and biological networks

- exponential growth in network theory in last 15 years
- algorithmic construction with given parameters

Social and biological networks

- exponential growth in network theory in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters

Social and biological networks

- exponential growth in network theory in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.

If $\exists$ simple graph $G(V, E)$ with $d(G) = \mathbf{d}$

$\Rightarrow$ $\mathbf{d}$ is a graphical sequence
G realizes $\mathbf{d}$.

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$
$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.
Question: how to decide whether **d** is graphical?

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$
$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.
Question: how to decide whether **d** is graphical?
Tutte's *f*-factor theorem (1949-52) (slow - not construct all)

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$
$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.

Question: how to decide whether **d** is graphical?

Tutte's f-factor theorem (1949-52) (slow - not construct all)
— Havel (1957) - Hakimi (1963) lemma

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$
$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.

Question: how to decide whether **d** is graphical?

Tutte's *f*-factor theorem (1949-52) (slow - not construct all)
— Havel (1957) - Hakimi (1963) lemma

Lemma: - any realization can be transformed by swaps into a canonical one

Degree sequences

$G(V;E)$ simple graph; $V=\{v_1,v_2,\ldots,v_n\}$ nodes
positive integers $\mathbf{d}=(d_1,d_2,\ldots,d_n)$.

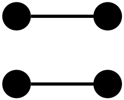
If $\exists$ simple graph $G(V,E)$ with $d(G)=\mathbf{d}$

$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.

Question: how to decide whether **d** is graphical?

Tutte's *f*-factor theorem (1949-52) (slow - not construct all)
— Havel (1957) - Hakimi (1963) lemma

Lemma: - any realization can be transformed by swaps into
a canonical one



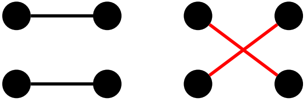
Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$
$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.

Question: how to decide whether **d** is graphical?

Tutte's f-factor theorem (1949-52) (slow - not construct all)
— Havel (1957) - Hakimi (1963) lemma

Lemma: - any realization can be transformed by swaps into
a canonical one



Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.

If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$

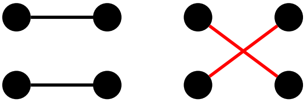
$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.

Question: how to decide whether **d** is graphical?

Tutte's f-factor theorem (1949-52) (slow - not construct all)

— Havel (1957) - Hakimi (1963) lemma

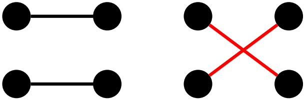
Lemma: - any realization can be transformed by swaps into
a canonical one



swap operation

Degree sequences

$G(V;E)$ simple graph; $V = \{v_1, v_2, \ldots, v_n\}$ nodes
positive integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$.
If $\exists$ simple graph $G(V,E)$ with $d(G) = \mathbf{d}$
$\Rightarrow$ **d** is a graphical sequence
G realizes **d**.
Question: how to decide whether **d** is graphical?
Tutte's f-factor theorem (1949-52) (slow - not construct all)
— Havel (1957) - Hakimi (1963) lemma
Lemma: - any realization can be transformed by swaps into
a canonical one



swap operation

Algorithm: - a greedy way to construct one realization (if $\exists$)

Havel's lemma vs. connection of realizations

Let G and H realizations of $\mathbf{d}$ Then

Havel's lemma vs. connection of realizations

Let G and H realizations of **d** Then
by Havel's lemma and via canonical realizations

Theorem ()

Exists swap-sequence for $G \longrightarrow H$.

Havel's lemma vs. connection of realizations

Let G and H realizations of $\mathbf{d}$ Then
by Havel's lemma and via canonical realizations

Theorem ()

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development

Havel's lemma vs. connection of realizations

Let G and H realizations of **d** Then
by Havel's lemma and via canonical realizations

Theorem (Petersen, 1891)

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development

Havel's lemma vs. connection of realizations

Let G and H realizations of **d** Then
by Havel's lemma and via canonical realizations

Theorem (Petersen, 1891)

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development
- the other direction is not trivial at all

Havel's lemma vs. connection of realizations

Let G and H realizations of $\mathbf{d}$ Then
by Havel's lemma (1957) and via canonical realizations

Theorem (Petersen, 1891)

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development
- the other direction is not trivial at all

Havel's lemma vs. connection of realizations

Let G and H realizations of $\mathbf{d}$ Then
by Havel's lemma (1957) and via canonical realizations

Theorem (Petersen, 1891)

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development
- the other direction is not trivial at all

Problem A, B, C vertex classes with $\mathbf{d}_A, \mathbf{d}_B, \mathbf{d}_C$ degree sequences. Looking for tripartite realizations

Havel's lemma vs. connection of realizations

Let G and H realizations of $\mathbf{d}$ Then
by Havel's lemma (1957) and via canonical realizations

Theorem (Petersen, 1891)

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development
- the other direction is not trivial at all

Problem A, B, C vertex classes with $\mathbf{d}_A, \mathbf{d}_B, \mathbf{d}_C$ degree sequences. Looking for tripartite realizations

- existence through Tutte's theorem is known

Havel's lemma vs. connection of realizations

Let G and H realizations of $\mathbf{d}$ Then
by Havel's lemma (1957) and via canonical realizations

Theorem (Petersen, 1891)

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development
- the other direction is not trivial at all

Problem A, B, C vertex classes with $\mathbf{d}_A, \mathbf{d}_B, \mathbf{d}_C$ degree sequences. Looking for tripartite realizations

- existence through Tutte's theorem is known
- with reasonable definitions $G \longrightarrow H$ is known

Havel's lemma vs. connection of realizations

Let G and H realizations of $\mathbf{d}$ Then
by Havel's lemma (1957) and via canonical realizations

Theorem (Petersen, 1891)

Exists swap-sequence for $G \longrightarrow H$.

- this is NOT a new development
- the other direction is not trivial at all

Problem A, B, C vertex classes with $\mathbf{d}_A, \mathbf{d}_B, \mathbf{d}_C$ degree sequences. Looking for tripartite realizations

- existence through Tutte's theorem is known
- with reasonable definitions $G \longrightarrow H$ is known
- there is NOT known Havel type greedy algorithm

Bipartite degree sequences

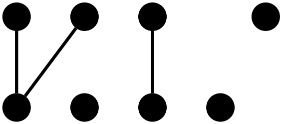
$G(U, W; E)$ simple bipartite graph, **bipartite d.s.**: $(\ell \leq k)$

$$\mathbf{bd}(G) = \Big((a_1, \ldots, a_k), (b_1, \ldots, b_\ell)\Big),$$

Bipartite degree sequences

$G(U, W; E)$ simple bipartite graph, **bipartite d.s.**: $(\ell \leq k)$

$$\mathbf{bd}(G) = \Big((a_1, \ldots, a_k), (b_1, \ldots, b_\ell) \Big),$$

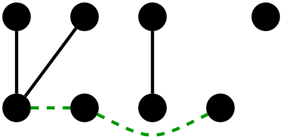


Bipartite degree sequences

$G(U, W; E)$ simple bipartite graph, **bipartite d.s.**: $(\ell \leq k)$

$$\mathbf{bd}(G) = \Big((a_1, \ldots, a_k), (b_1, \ldots, b_\ell)\Big),$$

- Forbidden edges

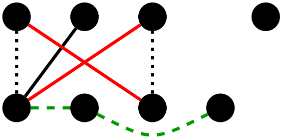


Bipartite degree sequences

$G(U, W; E)$ simple bipartite graph, **bipartite d.s.**: ($\ell \leq k$)

$$\mathbf{bd}(G) = \Big((a_1, \ldots, a_k), (b_1, \ldots, b_\ell)\Big),$$

- Forbidden edges
- $\exists$ swap operations (careful)



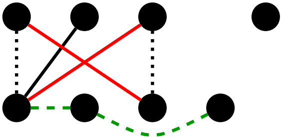
Bipartite degree sequences

$G(U, W; E)$ simple bipartite graph, **bipartite d.s.**: $(\ell \leq k)$

$$\mathbf{bd}(G) = \Big((a_1, \dots, a_k), (b_1, \dots, b_\ell)\Big),$$

- Forbidden edges
- $\exists$ swap operations (careful)

- Multigraphs - Long and venerable history



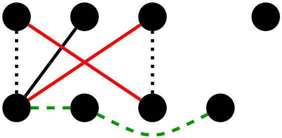
Bipartite degree sequences

$G(U, W; E)$ simple bipartite graph, **bipartite d.s.**: ($\ell \leq k$)

$$\mathbf{bd}(G) = \Big((a_1, \ldots, a_k), (b_1, \ldots, b_\ell) \Big),$$

- Forbidden edges
- $\exists$ swap operations (careful)

- Multigraphs - Long and venerable history
- Simple graphs There is HH-lemma and algorithm
  D.B. West's book (2001) and
  Kim - Toroczkai - Erdős - Miklós - Székely (2009)



Directed degree sequences

$\vec{G}(X;\vec{E})$ simple directed graph, $X = \{x_1, x_2, \ldots, x_n\}$

$$\mathbf{dd}(\vec{G}) = \Big((d_1^+, d_2^+, \ldots, d_n^+), (d_1^-, d_2^-, \ldots, d_n^-) \Big)$$

Directed degree sequences

$\vec{G}(X;\vec{E})$ simple directed graph, $X = \{x_1, x_2, \ldots, x_n\}$

$$\mathbf{dd}(\vec{G}) = \Big((d_1^+, d_2^+, \ldots, d_n^+), (d_1^-, d_2^-, \ldots, d_n^-) \Big)$$

- Directed multigraphs: $\exists$ HH lemma and algorithm

Directed degree sequences

$\vec{G}(X; \vec{E})$ simple directed graph, $X = \{x_1, x_2, \ldots, x_n\}$

$$\mathbf{dd}(\vec{G}) = \Big((d_1^+, d_2^+, \ldots, d_n^+), (d_1^-, d_2^-, \ldots, d_n^-) \Big)$$

- Directed multigraphs: $\exists$ HH lemma and algorithm
  Gale (1957), Ryser (1957), Hakimi (1963)

Directed degree sequences

$\vec{G}(X;\vec{E})$ simple directed graph, $X=\{x_1,x_2,\ldots,x_n\}$

$$\mathbf{dd}(\vec{G})=\Big(\left(d_1^+,d_2^+,\ldots,d_n^+\right),\left(d_1^-,d_2^-,\ldots,d_n^-\right)\Big)$$

- Directed multigraphs: $\exists$ HH lemma and algorithm
  Gale (1957), Ryser (1957), Hakimi (1963)
- Simple directed graphs: $\exists$ HH lemma and algorithm

Kleitman - Wang (1973) & Erdős - Miklós - Toroczkai (2010)

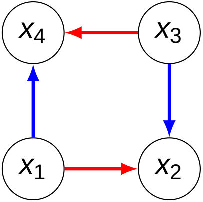
Directed degree sequences

$\vec{G}(X;\vec{E})$ simple directed graph, $X = \{x_1, x_2, \ldots, x_n\}$

$$\mathbf{dd}(\vec{G}) = \Big((d_1^+, d_2^+, \ldots, d_n^+), (d_1^-, d_2^-, \ldots, d_n^-) \Big)$$

- Directed multigraphs: $\exists$ HH lemma and algorithm
  Gale (1957), Ryser (1957), Hakimi (1963)
- Simple directed graphs: $\exists$ HH lemma and algorithm

Kleitman - Wang (1973) & Erdős - Miklós - Toroczkai (2010)

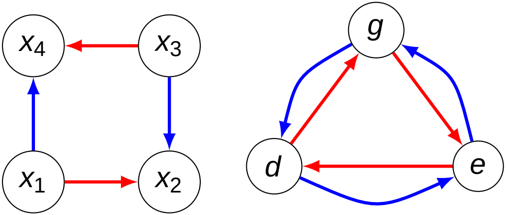


Directed degree sequences

$\vec{G}(X;\vec{E})$ simple directed graph, $X = \{x_1, x_2, \ldots, x_n\}$

$$\mathbf{dd}(\vec{G}) = \Big((d_1^+, d_2^+, \ldots, d_n^+), (d_1^-, d_2^-, \ldots, d_n^-) \Big)$$

- Directed multigraphs: $\exists$ HH lemma and algorithm
  Gale (1957), Ryser (1957), Hakimi (1963)
- Simple directed graphs: $\exists$ HH lemma and algorithm
  Kleitman - Wang (1973) & Erdős - Miklós - Toroczkai (2010)



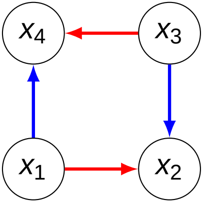
Directed degree sequences

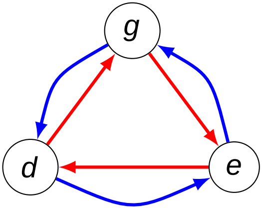
$\vec{G}(X;\vec{E})$ simple directed graph, $X = \{x_1, x_2, \ldots, x_n\}$

$$\mathbf{dd}(\vec{G}) = \Big((d_1^+, d_2^+, \ldots, d_n^+), (d_1^-, d_2^-, \ldots, d_n^-) \Big)$$

- Directed multigraphs: $\exists$ HH lemma and algorithm
  Gale (1957), Ryser (1957), Hakimi (1963)
- Simple directed graphs: $\exists$ HH lemma and algorithm

Kleitman - Wang (1973) & Erdős - Miklós - Toroczkai (2010)





$$\vec{de}, \vec{gd}, \vec{eg}$$
$$\Updownarrow$$
$$\vec{ed}, \vec{dg}, \vec{ge}$$

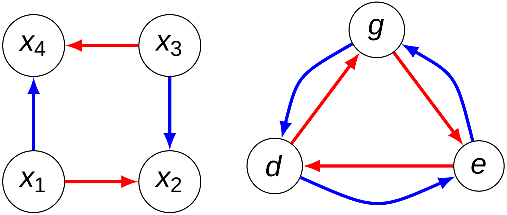
Directed degree sequences

$\vec{G}(X;\vec{E})$ simple directed graph, $X = \{x_1, x_2, \ldots, x_n\}$

$$\mathbf{dd}(\vec{G}) = \Big((d_1^+, d_2^+, \ldots, d_n^+), (d_1^-, d_2^-, \ldots, d_n^-) \Big)$$

- Directed multigraphs: ∃ HH lemma and algorithm
  Gale (1957), Ryser (1957), Hakimi (1963)
- Simple directed graphs: ∃ HH lemma and algorithm

Kleitman - Wang (1973) & Erdős - Miklós - Toroczkai (2010)

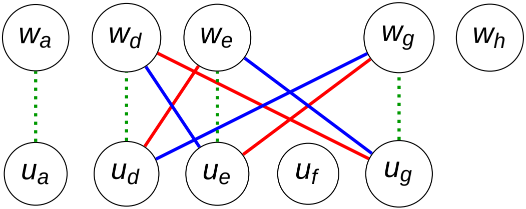


$\vec{de}, \vec{gd}, \vec{eg}$
$\Updownarrow$
$\vec{ed}, \vec{dg}, \vec{ge}$

With reasonable definitions: ∃ HH lemma and $\vec{G} \longrightarrow \vec{H}$ algorithm via directed swaps

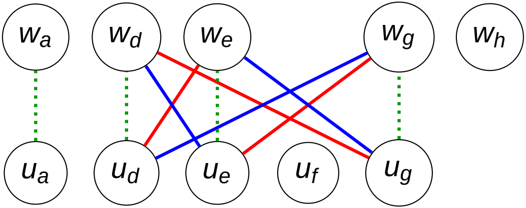
Representing directed graphs (Gale 1957)

with the bipartite graph $B(\vec{G}) = (U, W; E)$
$u_i \in U$ - out-edges from $v_i \in W$ in-edges to x_i.



Representing directed graphs (Gale 1957)

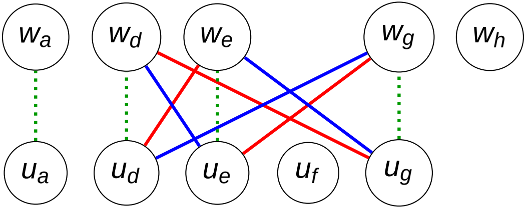
with the bipartite graph $B(\vec{G}) = (U, W; E)$
$u_i \in U$ - out-edges from $v_i \in W$ in-edges to x_i.



There are forbidden edges

e.g. $u_a w_a, \dots, u_g w_g$

with the bipartite graph $B(\vec{G}) = (U, W; E)$
$u_i \in U$ - out-edges from $v_i \in W$ in-edges to x_i.



There are forbidden edges
e.g. $u_a w_a, \ldots, u_g w_g$
the usual swaps between $B(\vec{G})$ and $B(\vec{H})$ represent
directed swaps between $\vec{G}$ and $\vec{H}$

Problem description

given degree sequence **d**; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

Problem description

given degree sequence $\mathbf{d}$; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph $G \,:\, d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Problem description

given degree sequence $\mathbf{d}$; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph G : $d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Solution: Tutte's f-factor theorem (1952) for $K_n \setminus \mathcal{F}$

Problem description

given degree sequence $\mathbf{d}$; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph G : $d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Solution: Tutte's f-factor theorem (1952) for $K_n \setminus \mathcal{F}$
provides a polynomial algorithm to decide the existence

Problem description

given degree sequence **d**; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph $G \ : \ d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Solution: Tutte's f-factor theorem (1952) for $K_n \setminus \mathcal{F}$
provides a polynomial algorithm to decide the existence

with reasonable definitions exists $G \rightarrow H$ swap-sequence

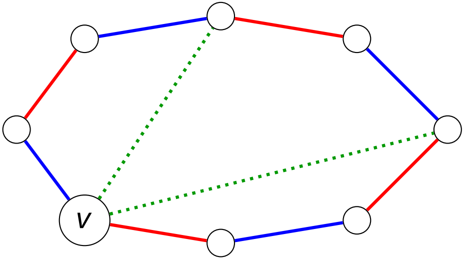
Problem description

given degree sequence $\mathbf{d}$; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph G : $d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Solution: Tutte's f-factor theorem (1952) for $K_n \setminus \mathcal{F}$
provides a polynomial algorithm to decide the existence

with reasonable definitions exists $G \rightarrow H$ swap-sequence



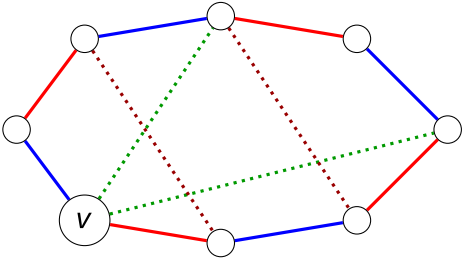
Problem description

given degree sequence $\mathbf{d}$; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph $G \ : \ d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Solution: Tutte's f-factor theorem (1952) for $K_n \setminus \mathcal{F}$
provides a polynomial algorithm to decide the existence

with reasonable definitions exists $G \rightarrow H$ swap-sequence



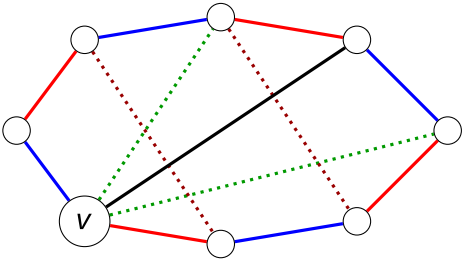
Problem description

given degree sequence $\mathbf{d}$; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph $G \ : \ d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Solution: Tutte's f-factor theorem (1952) for $K_n \setminus \mathcal{F}$
provides a polynomial algorithm to decide the existence

with reasonable definitions exists $G \rightarrow H$ swap-sequence



Problem description

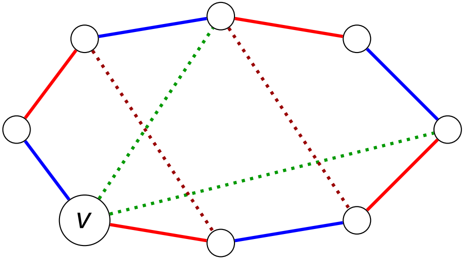
given degree sequence $\mathbf{d}$; $\mathcal{F} \subset \binom{V}{2}$ of forbidden edges

The restricted degree sequence problem $\mathbf{d}^{\mathcal{F}}$:
$\exists$? simple graph G : $d(G) = \mathbf{d}$ which completely avoids $\mathcal{F}$

Solution: Tutte's f-factor theorem (1952) for $K_n \setminus \mathcal{F}$
provides a polynomial algorithm to decide the existence

with reasonable definitions exists $G \rightarrow H$ swap-sequence

$\exists$ an $\mathcal{F}$-swap



Examples for $\mathcal{F}$-compatible swaps

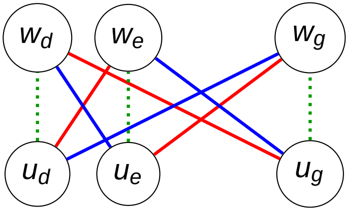
circular C_4 $\mathcal{F}$-swap = Havel–Hakimi swap.

Examples for $\mathcal{F}$-compatible swaps

circular C_4 $\mathcal{F}$-swap = Havel–Hakimi swap.
circular C_6 $\mathcal{F}$-swap = triangular C_6-swap,

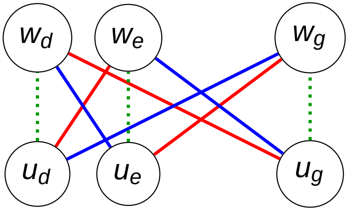
Examples for $\mathcal{F}$-compatible swaps

circular C_4 $\mathcal{F}$-swap = Havel–Hakimi swap.
circular C_6 $\mathcal{F}$-swap = triangular C_6-swap,



Examples for $\mathcal{F}$-compatible swaps

circular C_4 $\mathcal{F}$-swap = Havel–Hakimi swap.
circular C_6 $\mathcal{F}$-swap = triangular C_6-swap,

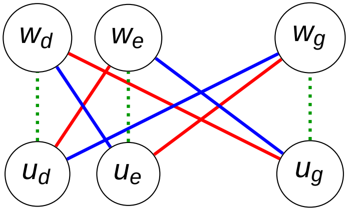


Theorem

G, H realizations of $\mathbf{d}^{\mathcal{F}}$ then $\exists G \to H$ with $\mathcal{F}$-swaps

Examples for $\mathcal{F}$-compatible swaps

circular C_4 $\mathcal{F}$-swap = Havel–Hakimi swap.
circular C_6 $\mathcal{F}$-swap = triangular C_6-swap,



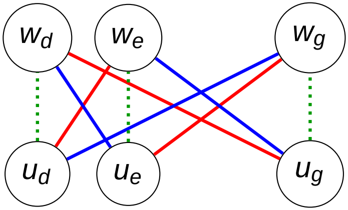
Theorem

G, H realizations of $\mathbf{d}^{\mathcal{F}}$ then $\exists G \to H$ with $\mathcal{F}$-swaps

Examples - directed graphs - connected, with Havel's lemma

Examples for $\mathcal{F}$-compatible swaps

circular C_4 $\mathcal{F}$-swap = Havel–Hakimi swap.
circular C_6 $\mathcal{F}$-swap = triangular C_6-swap,



Theorem

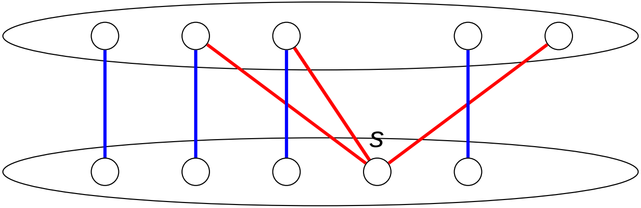
G, H realizations of $\mathbf{d}^{\mathcal{F}}$ then $\exists G \to H$ with $\mathcal{F}$-swaps

Examples - directed graphs - connected, with Havel's lemma
tripartite graphs - connected, no Havel's lemma

A simple example: star+factor problem

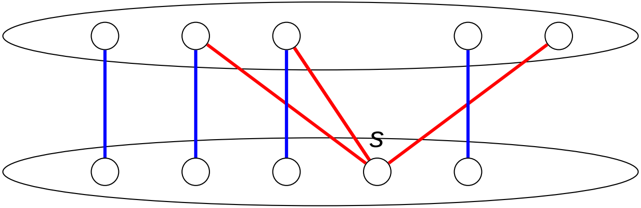
In its simplest form:
d is bipartite, and $\mathcal{F} =$ union of 1-factor and a star

In its simplest form:
d is bipartite, and $\mathcal{F} =$ union of 1-factor and a star



A simple example: star+factor problem

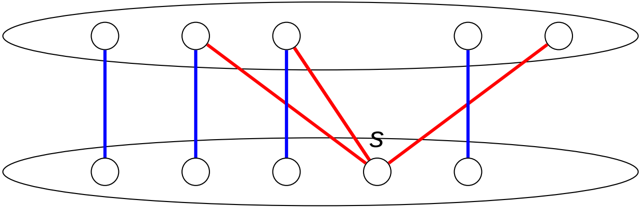
In its simplest form:
d is bipartite, and $\mathcal{F} =$ union of 1-factor and a star



Tutte's *f*-factor theorem applies

A simple example: star+factor problem

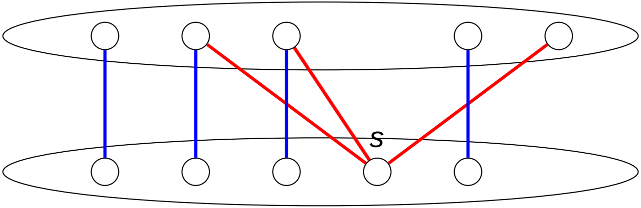
In its simplest form:
d is bipartite, and $\mathcal{F} =$ union of 1-factor and a star



Tutte's f-factor theorem applies
realizations are connected

A simple example: star+factor problem

In its simplest form:
d is bipartite, and $\mathcal{F} =$ union of 1-factor and a star



Tutte's f-factor theorem applies
realizations are connected
there exists a Havel-type approach

1 Background

2 Restricted degree sequences

3 Application: counting realizations of $\mathbf{d}^{\mathcal{F}}$

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

A classical example

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

A classical example

epidemics studies of sexually transmitted diseases

Liljeros F, Edling C R, Amaral L A N, Stanley H E and Åberg Y 2001 *Nature* **411**

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

A classical example

epidemics studies of sexually transmitted diseases

Liljeros F, Edling C R, Amaral L A N, Stanley H E and Åberg Y 2001 *Nature* **411**

- data collected is from anonymous surveys

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

A classical example

epidemics studies of sexually transmitted diseases

Liljeros F, Edling C R, Amaral L A N, Stanley H E and Åberg Y 2001 *Nature* **411**

- data collected is from anonymous surveys
- number of different partners in a given period of time,
- without revealing their identity.

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

A classical example

epidemics studies of sexually transmitted diseases

Liljeros F, Edling C R, Amaral L A N, Stanley H E and Åberg Y 2001 *Nature* **411**

- data collected is from anonymous surveys
- number of different partners in a given period of time,
- without revealing their identity.
- constructing the most typical contact graph
- obeying the empirical degree sequence.

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

An other ancient examples

J. K. Senior: Partitions and their Representative Graphs, *Amer. J. Math.*, **73** (1951), 663–689.

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

An other ancient examples

J. K. Senior: Partitions and their Representative Graphs, *Amer. J. Math.*, **73** (1951), 663–689.

- find all possible molecules with given composition but with different structures

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

An other ancient examples

J. K. Senior: Partitions and their Representative Graphs, *Amer. J. Math.*, **73** (1951), 663–689.

- find all possible molecules with given composition but with different structures
- generating all possible graphs with multiple edges but no loops

Constrcuting and counting realizations

Applied network theory: exponential growth in last 15 years
- algorithmic construction with given parameters
- uniform sampling all networks with that given parameters
- (approximate) counting of all instances

An other ancient examples

J. K. Senior: Partitions and their Representative Graphs, *Amer. J. Math.*, **73** (1951), 663–689.

- find all possible molecules with given composition but with different structures
- generating all possible graphs with multiple edges but no loops
- introduced swaps (but called transfusion)

Sampling and counting realizations

Goal: to find a typical or random realization

Sampling and counting realizations

Goal: to find a typical or random realization
Markov Chain Monte Carlo (MCMC) methods

Sampling and counting realizations

Goal: to find a typical or random realization
Markov Chain Monte Carlo (MCMC) methods

start with an arbitrary realization and take a long enough series of randomly chosen $G \longrightarrow H$

Sampling and counting realizations

Goal: to find a typical or random realization
Markov Chain Monte Carlo (MCMC) methods

start with an arbitrary realization and take a long enough series of randomly chosen $G \longrightarrow H$

- this method always produces a random realization!

Sampling and counting realizations

Goal: to find a typical or random realization
Markov Chain Monte Carlo (MCMC) methods

start with an arbitrary realization and take a long enough series of randomly chosen $G \longrightarrow H$

- this method always produces a random realization!

BUT how fast ????

Sampling and counting realizations

Goal: to find a typical or random realization
Markov Chain Monte Carlo (MCMC) methods

start with an arbitrary realization and take a long enough series of randomly chosen $G \longrightarrow H$

- this method always produces a random realization!
  BUT how fast ????
- considered fast enough if it is polynomial in time
  and called fast mixing

Sampling and counting realizations

Goal: to find a typical or random realization
Markov Chain Monte Carlo (MCMC) methods

start with an arbitrary realization and take a long enough series of randomly chosen $G \longrightarrow H$

- this method always produces a random realization!
  BUT how fast ????
- considered fast enough if it is polynomial in time
  and called fast mixing

Theorem (Jerrum, Valiant and Vazirani (1986))

if the problem is Self-reducible then fast mixing MCMC sampling provides a good approximation on the number of realizations

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite
- Greenhill (2011) regular directed graphs – equivalent with regular bipartite **d** with a forbidden one-factor $\mathcal{F}$

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite
- Greenhill (2011) regular directed graphs – equivalent with regular bipartite **d** with a forbidden one-factor $\mathcal{F}$
- all are fast mixing but NOT self-reducible

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite
- Greenhill (2011) regular directed graphs – equivalent with regular bipartite **d** with a forbidden one-factor $\mathcal{F}$
- all are fast mixing but NOT self-reducible
- Erdős-Kiss-Miklós-Soukup (2014?) - half-regular, bipartite **d** with forbidden star+factor

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite
- Greenhill (2011) regular directed graphs – equivalent with regular bipartite **d** with a forbidden one-factor $\mathcal{F}$
- all are fast mixing but NOT self-reducible
- Erdős-Kiss-Miklós-Soukup (2014?) - half-regular, bipartite **d** with forbidden star+factor
  - fast mixing

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite
- Greenhill (2011) regular directed graphs – equivalent with regular bipartite **d** with a forbidden one-factor $\mathcal{F}$
- all are fast mixing but NOT self-reducible
- Erdős-Kiss-Miklós-Soukup (2014?) - half-regular, bipartite **d** with forbidden star+factor
  - fast mixing
  - contains the above results

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite
- Greenhill (2011) regular directed graphs – equivalent with regular bipartite **d** with a forbidden one-factor $\mathcal{F}$
- all are fast mixing but NOT self-reducible
- Erdős-Kiss-Miklós-Soukup (2014?) - half-regular, bipartite **d** with forbidden star+factor
  - fast mixing
  - contains the above results
  - self-reducible

Examples for fast mixing MCMC on realizations

Examples:

- Kannan-Tetali-Vempala (1999) - **d** is regular bipartite
- Miklós-Erdős-Soukup (2013) - **d** half-regular bipartite
- Greenhill (2011) regular directed graphs – equivalent with regular bipartite **d** with a forbidden one-factor $\mathcal{F}$
- all are fast mixing but NOT self-reducible
- Erdős-Kiss-Miklós-Soukup (2014?) - half-regular, bipartite **d** with forbidden star+factor
  - fast mixing
  - contains the above results
  - self-reducible
- all MCMC above are suitable for approximate counting