Clumsy Packings with Polyominoes

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packed packings

(many applications,
papers, results . . .)

clumsy packings

(very little known,
This talk . . .)

theworststuffever.com

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(many applications,
papers, results . . .)

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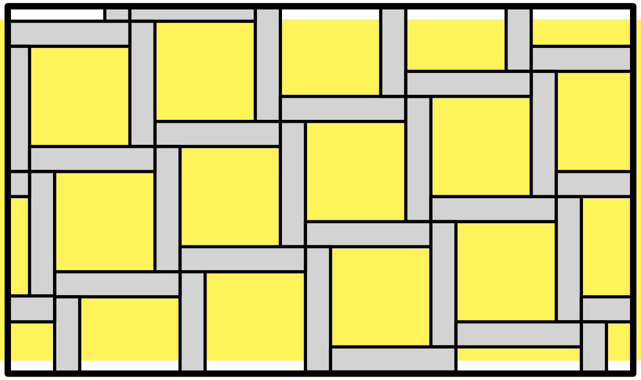
Clumsy Packings with Polyominoes



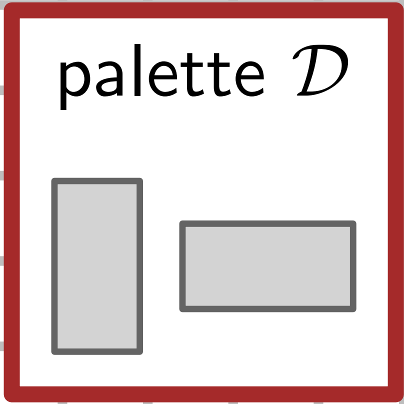
I. Introduction

- Definitions & Examples

II. Results

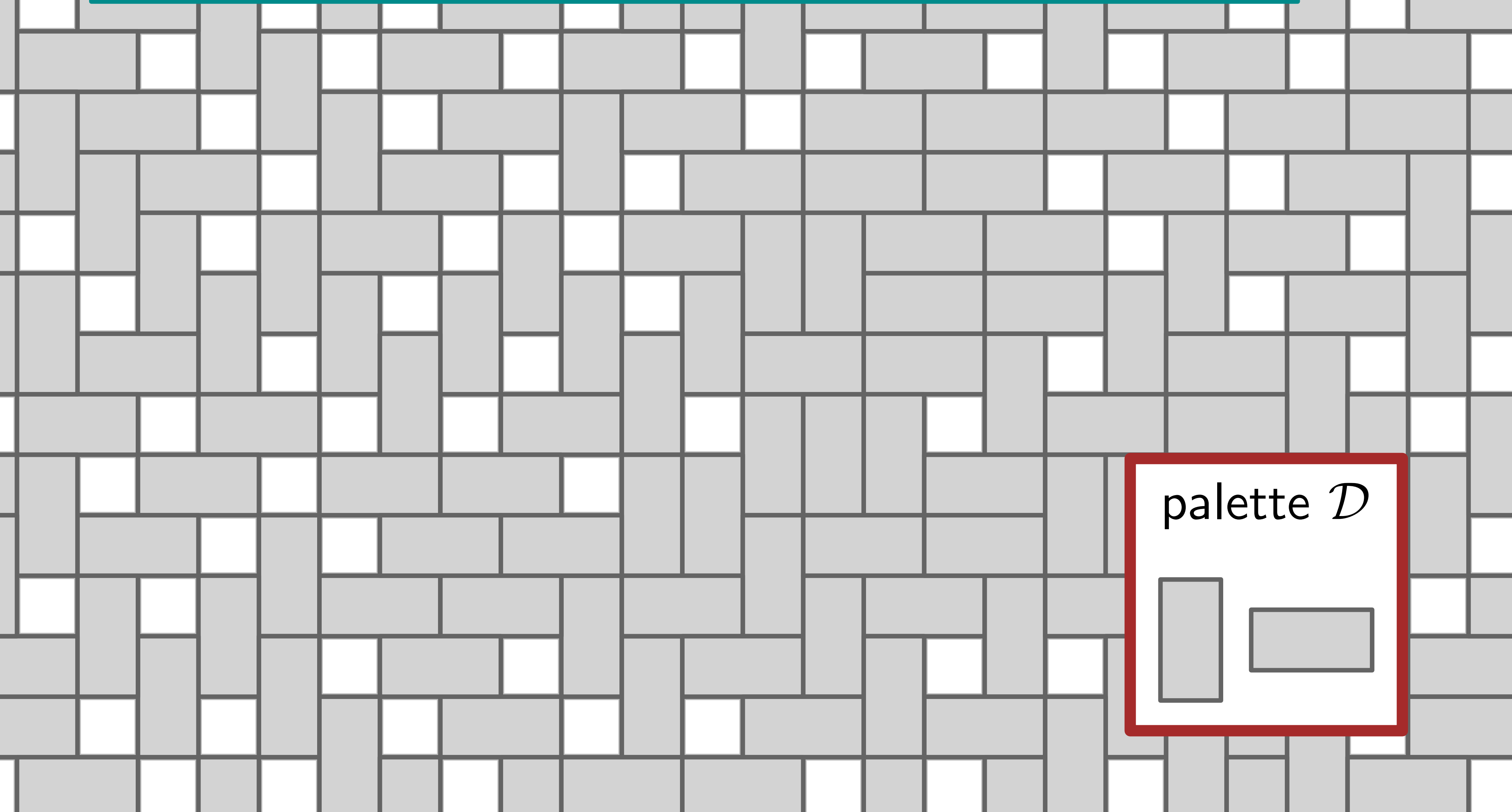
- Extremal Questions
- Aperiodic Clumsy Packings
- Undecidability

Example & Definitions



Example & Definitions

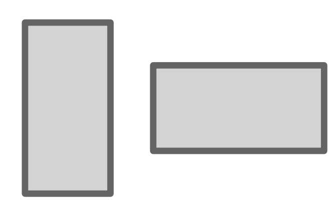
packing P = maximal set of disjoint copies from $\mathcal{D}$



Example & Definitions

- **packing** P = maximal set of disjoint copies from $\mathcal{D}$
- **density**$(P) = \lim_{n\to\infty} \frac{\text{area covered by } P \text{ in } n\text{-ball}}{\text{area of } n\text{-ball}}$
- **clumsiness**$(\mathcal{D})$ = minimum density of a packing

palette $\mathcal{D}$



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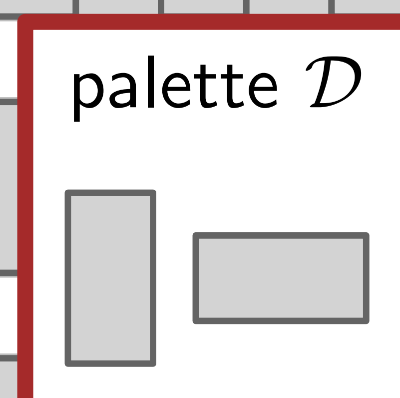

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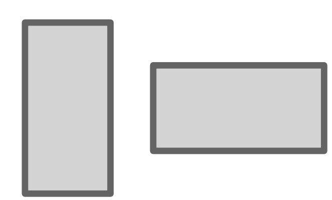
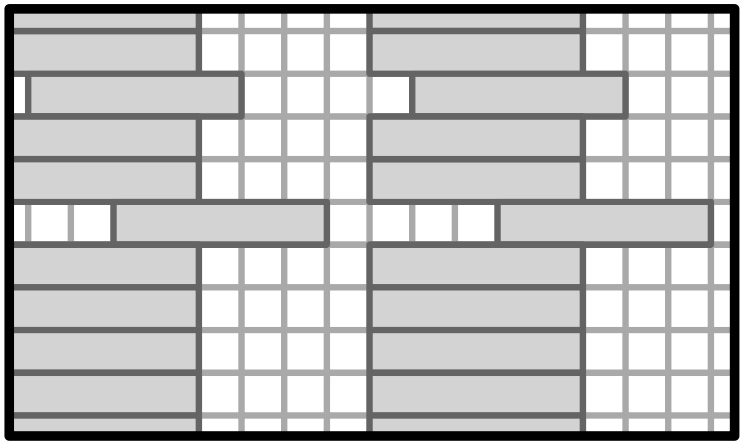
- **Thm.**(Gyárfás, Lehel, Tuza 1988)

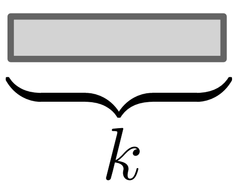
  clumsiness([vertical rectangle] , [horizontal rectangle]) $= 2/3.$

palette $\mathcal{D}$

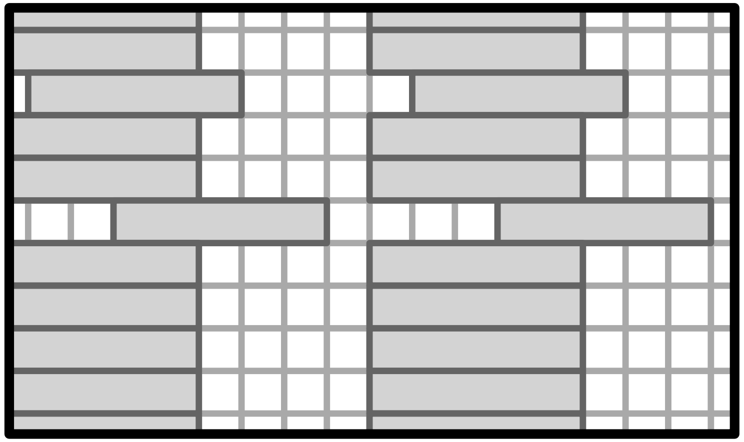


More Examples

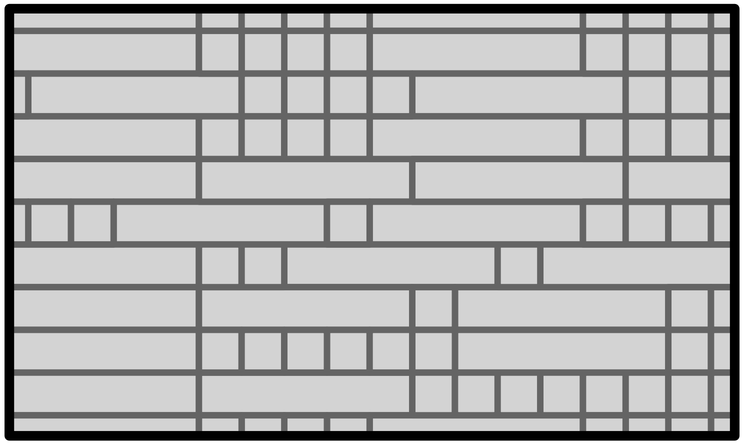


clumsiness() $= \frac{k}{2k-1} \approx 1/2$

More Examples



$$\text{clumsiness}\left(\underbrace{\square}_{k}\right) = \frac{k}{2k-1} \approx 1/2$$



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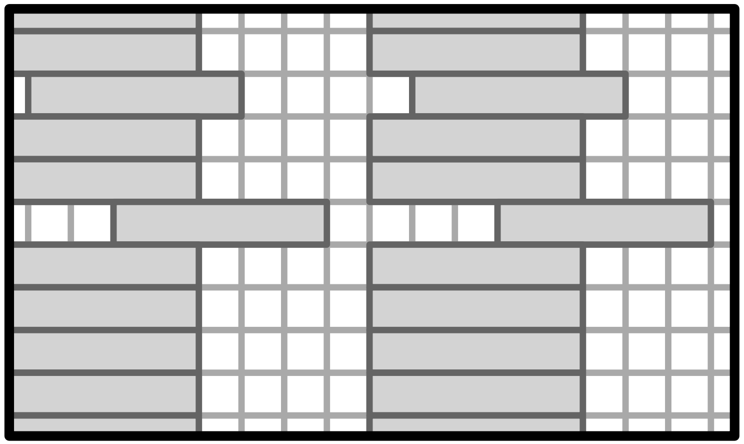
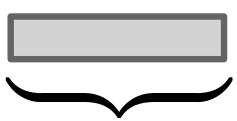
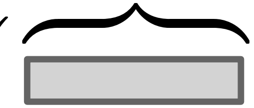
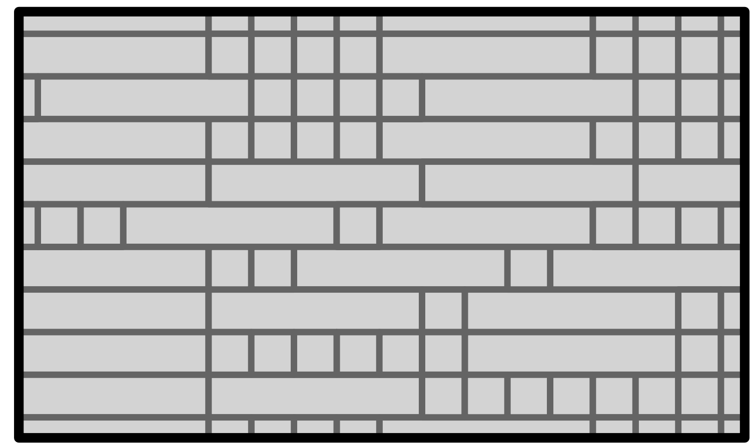
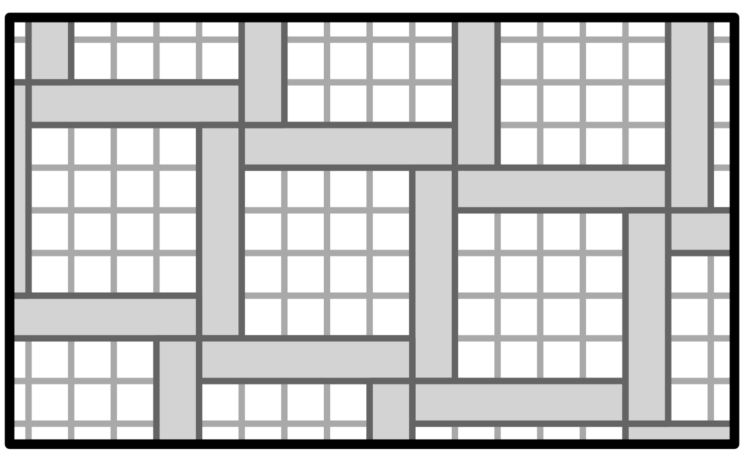


clumsiness( k) $= \frac{k}{2k-1} \approx 1/2$

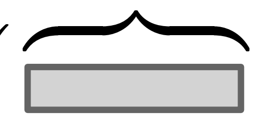
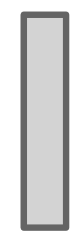
k

clumsiness( , ) $= 1$





k

clumsiness( , )

$\leq \frac{2k}{k(k-1)+1} \approx 2/k$

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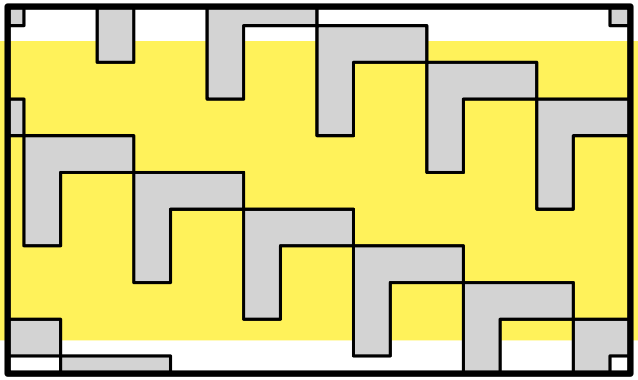
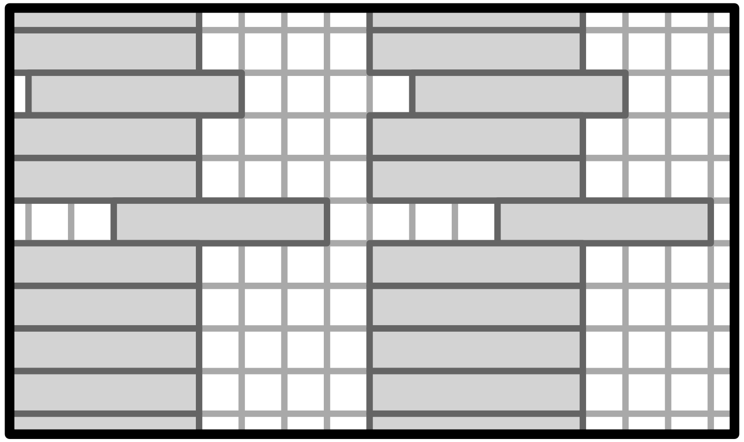


The Clumsiest Polyomino

What is the smallest clumsiness
if $\mathcal{D}$ is a single polyomino of size k?



clumsiness$\Big($ $\overbrace{\text{▭}}^{k}$ $\Big)$

$= \frac{k}{2k-1} \approx$ $1/2$

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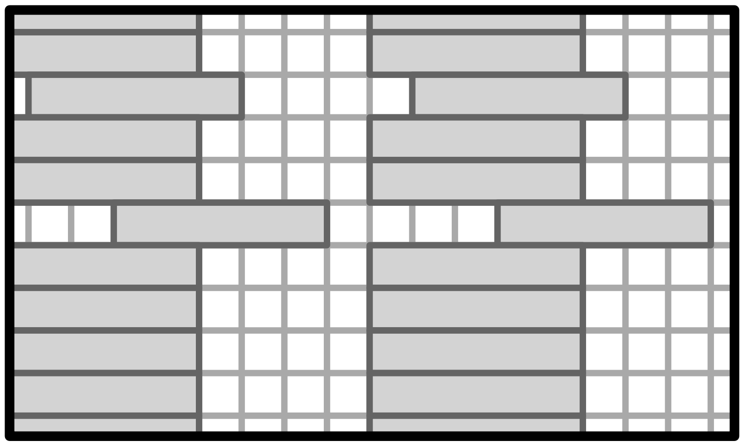
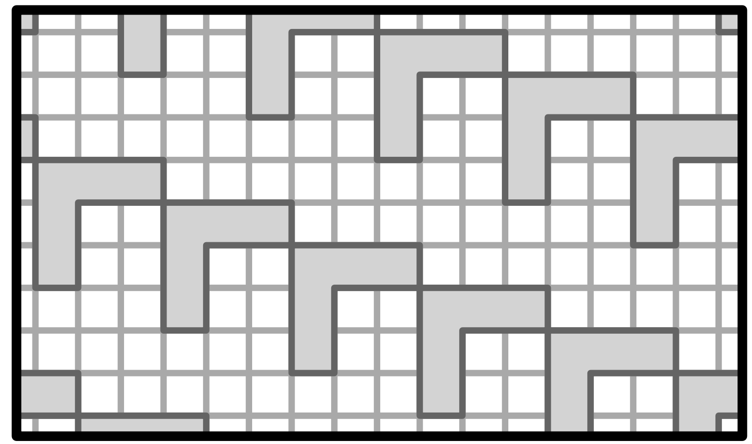
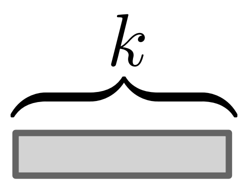
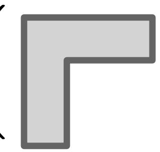




clumsiness()

$$= \frac{k}{2k-1} \approx \boxed{1/2}$$

clumsiness()

$$= \frac{k}{k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2} \approx \boxed{2/k}$$

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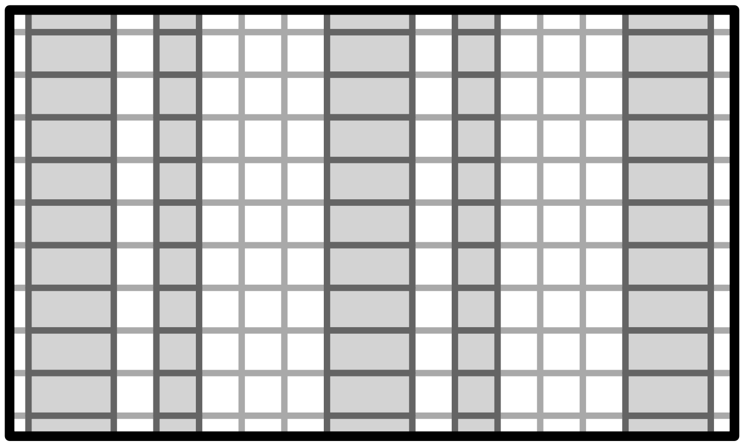

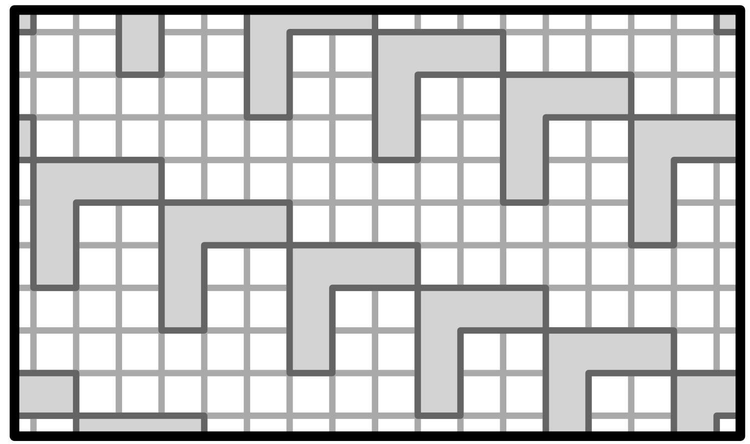

clumsiness()

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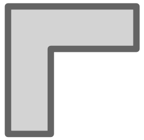
clumsiness()

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- **Theorem.** Let D be a single polyomino of size k. Then
  - ▷ clumsiness$\big(D\big) \geq k/(k^2 - k + 1)$.
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  Both bounds are best possible.

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- **Lemma.** Let D be a single polyomino of size k. Then
  - ▷ D intersects $\leq k^2 - k + 1$ copies of itself. $\approx k^2$
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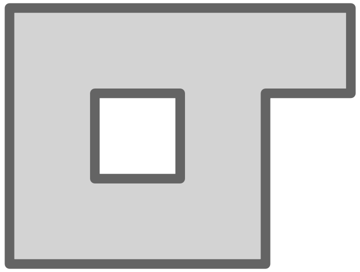
Proof of Lemma

- **Lemma.** A polyomino of size k intersects at most $k^2 - k + 1$ copies of itself (including itself).

fix D



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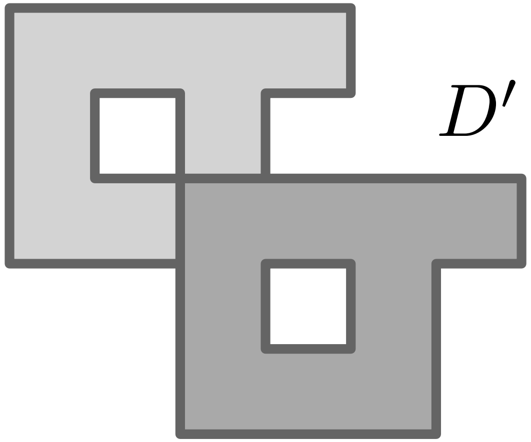

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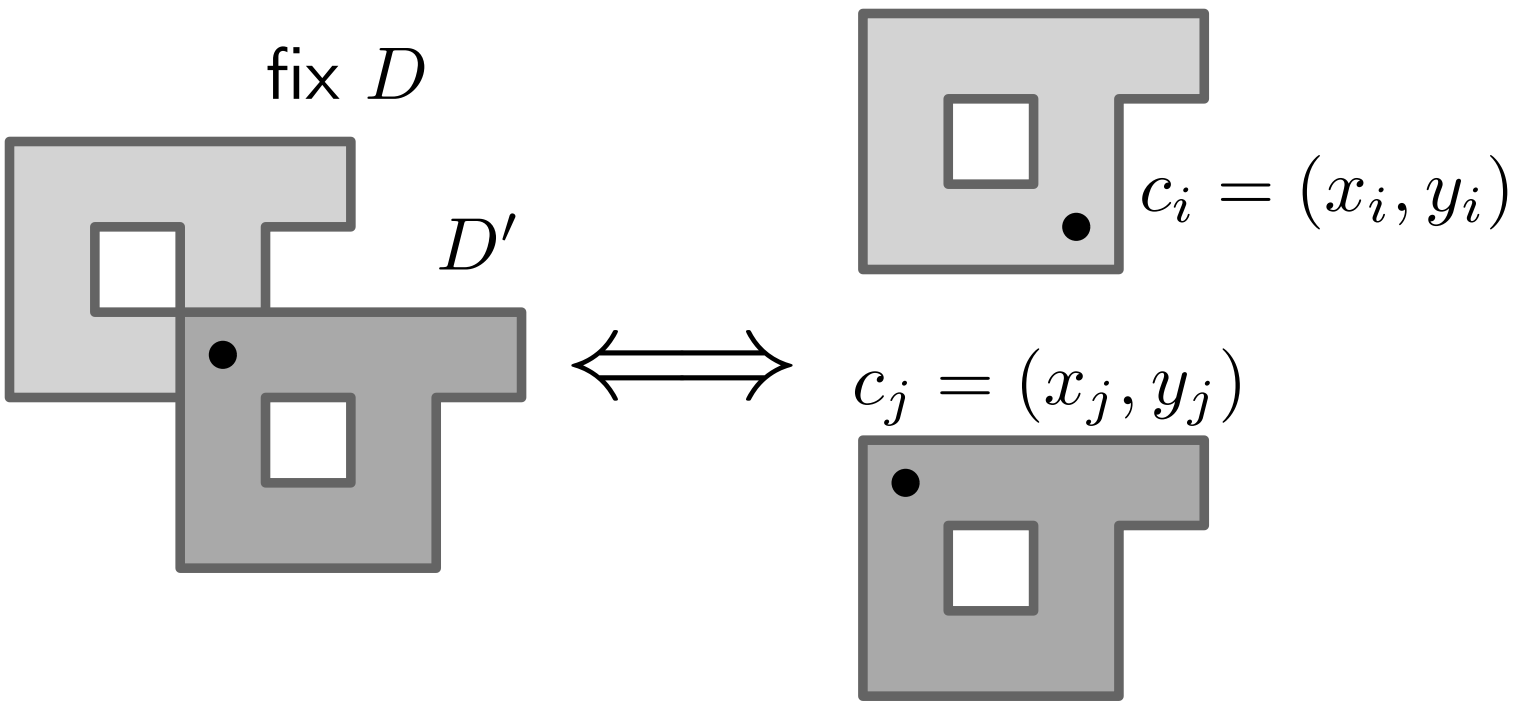

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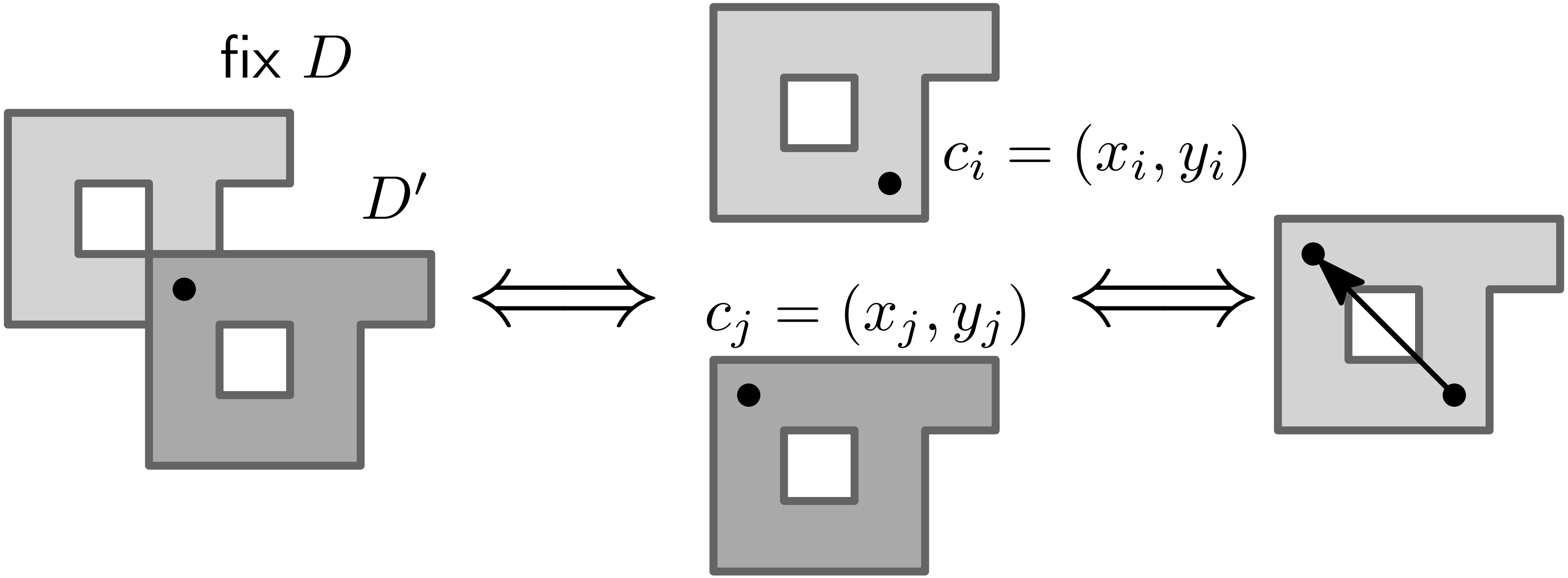
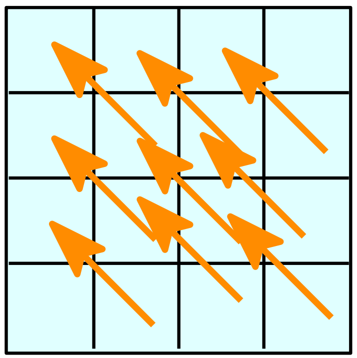


D' intersects D $\iff$ cell c_i of D coincides with cell c_j of D' $\iff$ $c_i - c_j$ is a difference in D

Examles of intersections of polyominoes with themselves.



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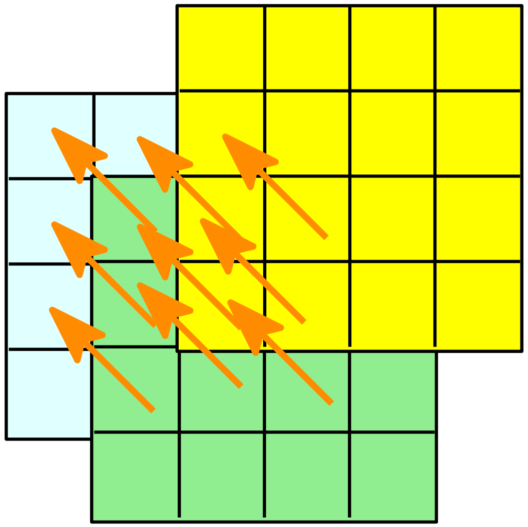

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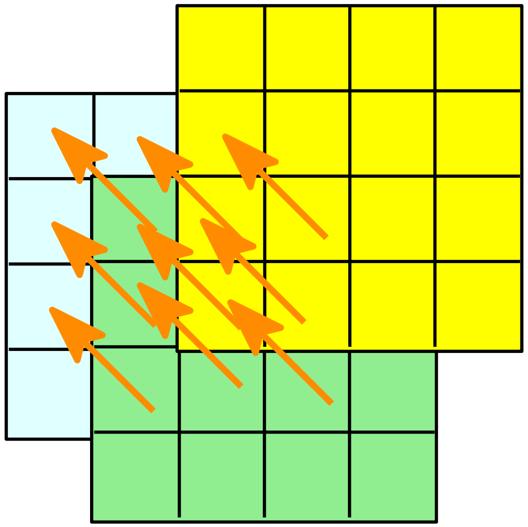

1 2 3 4 5 6 7 8 9 10 11 12 13

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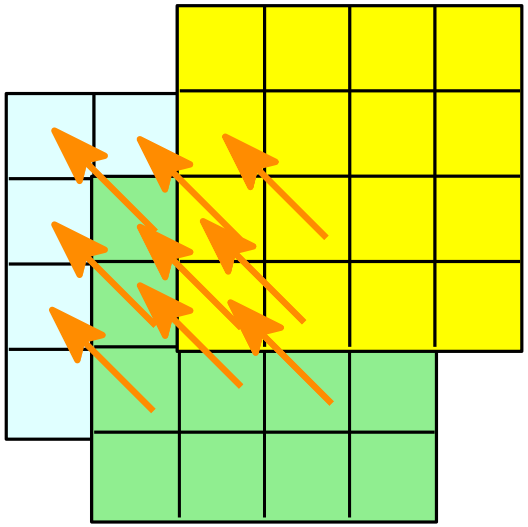

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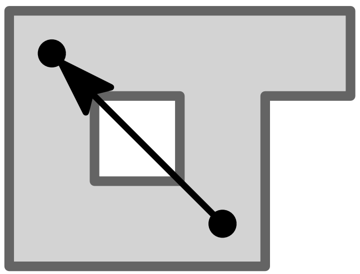
Proof of Lemma (general case)

Lemma. In a polyomino of size k there are at most $k^2 - k + 1$ distinct differences.



- There are k^2 differences $c_i - c_j$.
- The k differences $c_i - c_i$ are the same.

Thm. (Singer 1938) For every prime power n there is a set S_n of $n+1$ numbers in $\mathbb{Z}_{n^2+n+1}$ such that every non-zero difference modulo n^2+n+1 occurs exactly once in S_n.

E.g. $S_2 = \{1, 2, 4\}$ and $S_3 = \{1, 2, 4, 8, 13\}$

5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1	2	3	4	5	6	0	1

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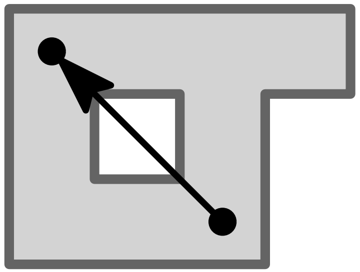

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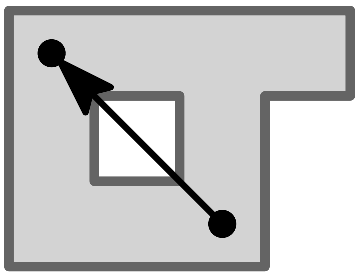

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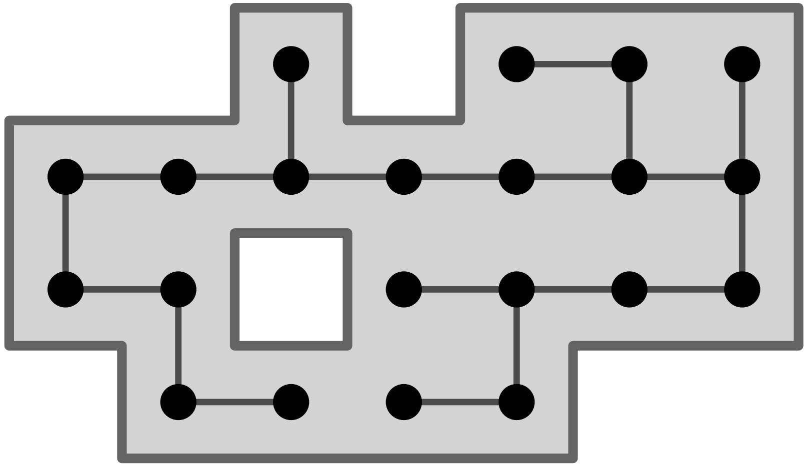
Lemma. A connected polyomino of size k has at most $k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2$ distinct differences.



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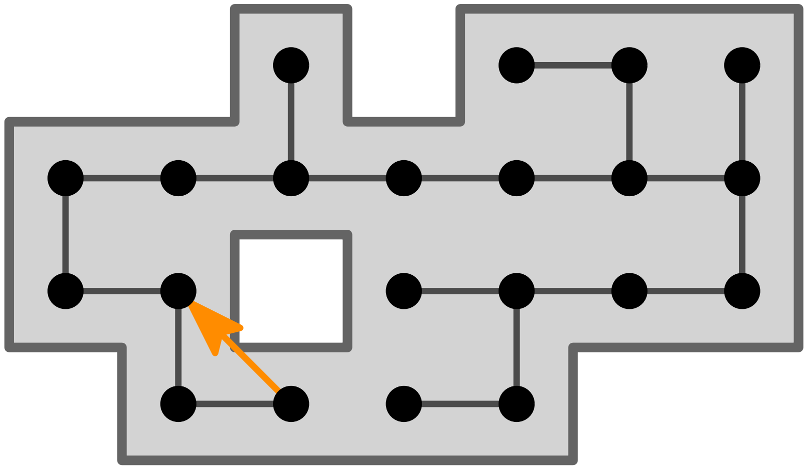

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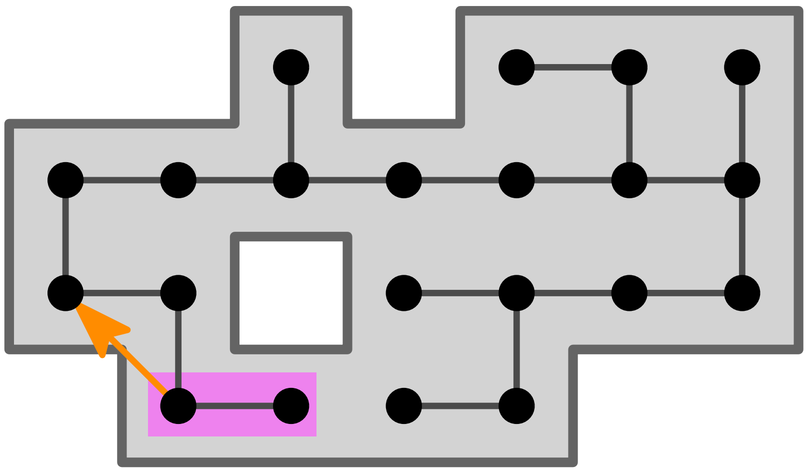

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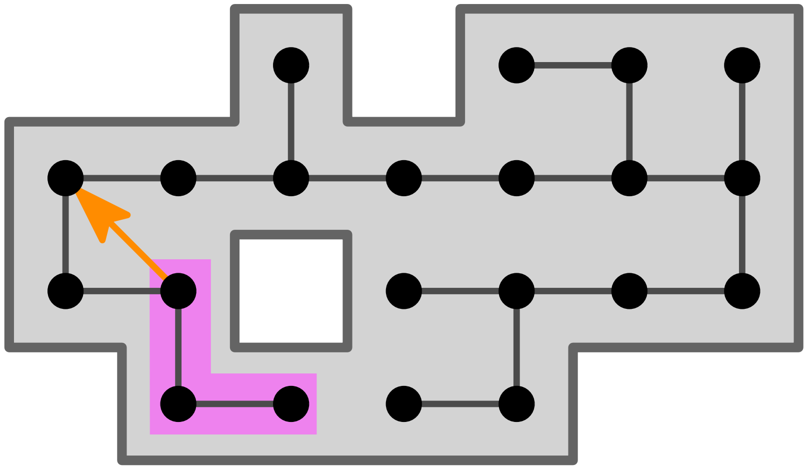

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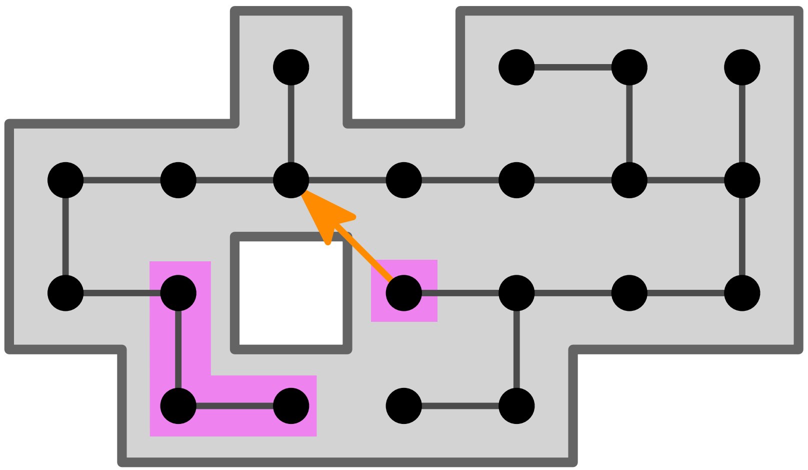

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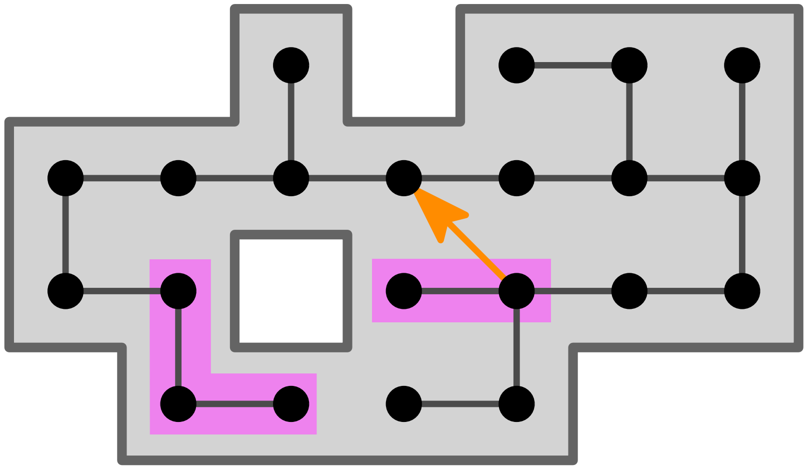

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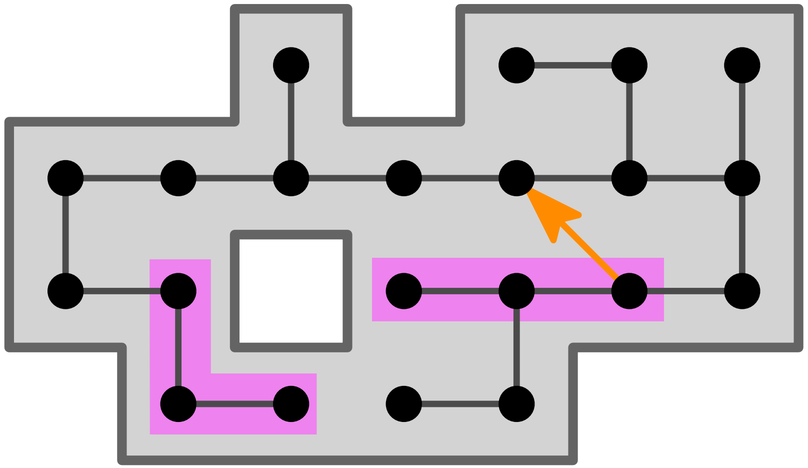

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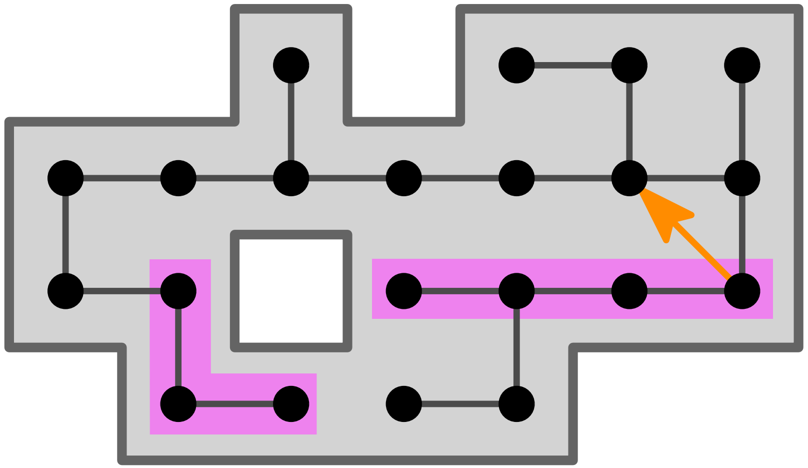

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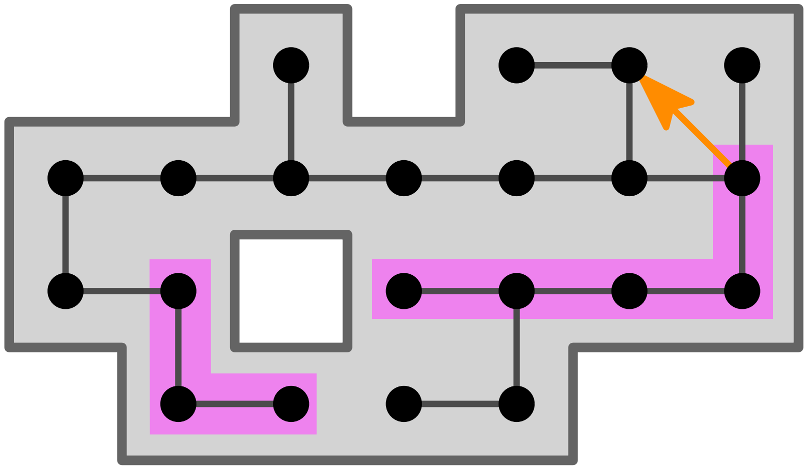

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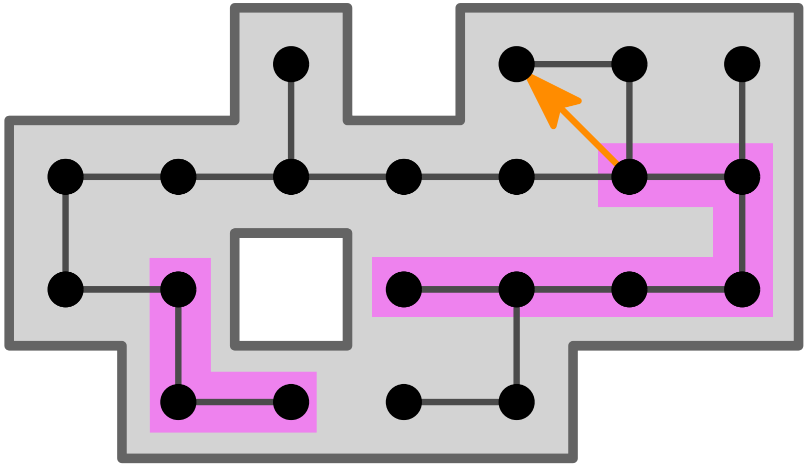
Proof of Lemma (connected case)

Lemma. A connected polyomino of size k has at most $k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2$ distinct differences.



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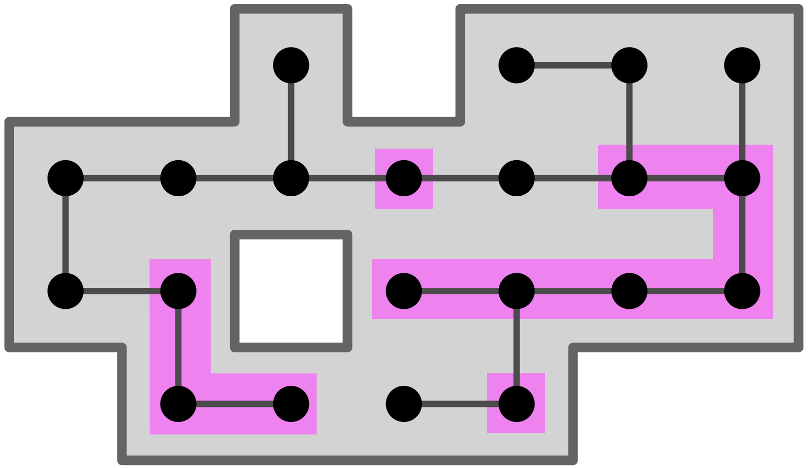

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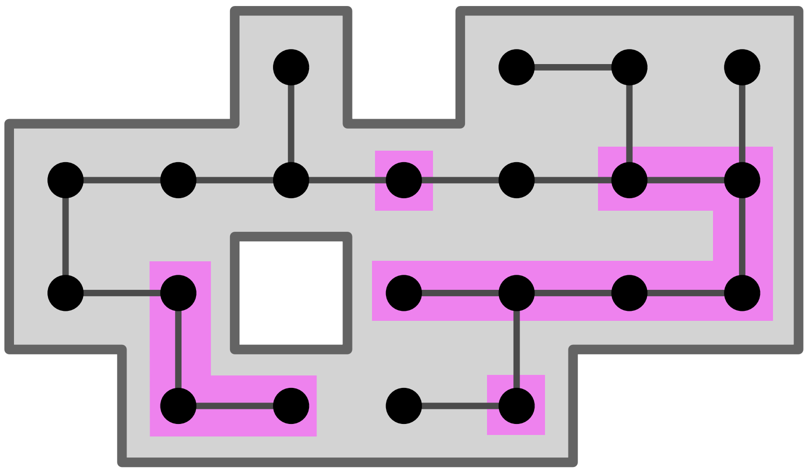

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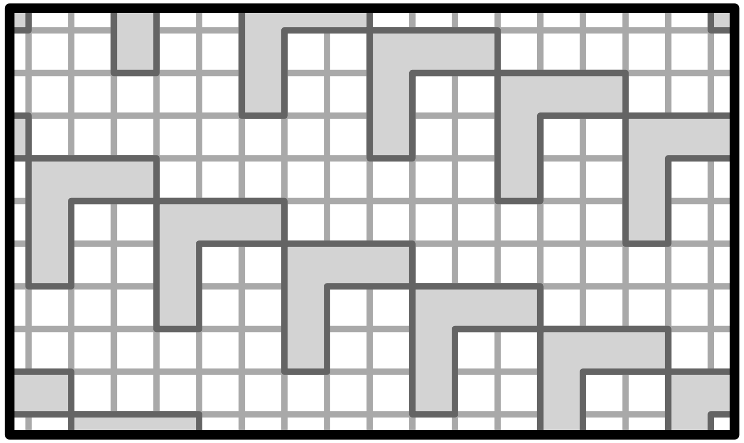
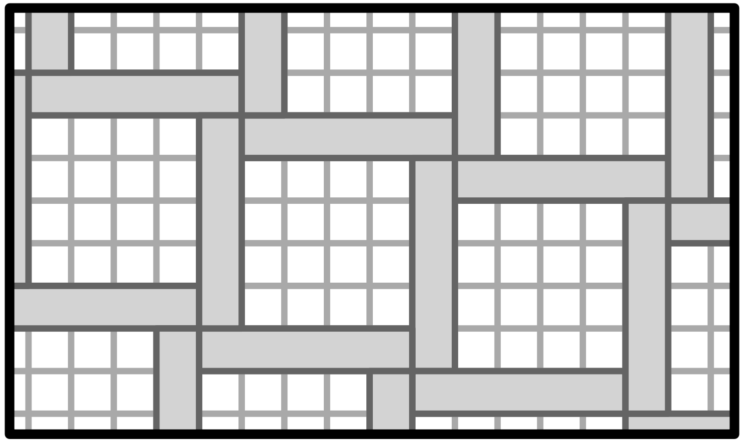
$\#$ differences $\leq k^2 - \#($  $)^2 - \#($  $)^2$

The Clumsiest Set of Polyominoes

- The clumsiest polyomino of size k has clumsiness $\frac{k}{k^2 - \lfloor (k-1)/2 \rfloor^2 - \lceil (k-1)/2 \rceil^2} \approx 2/k$.





clumsiness $\leq \frac{2k}{k(k-1)+1} \approx 2/k$

- **Open Question:** What is the clumsiest set of polyominoes each of size at most k?

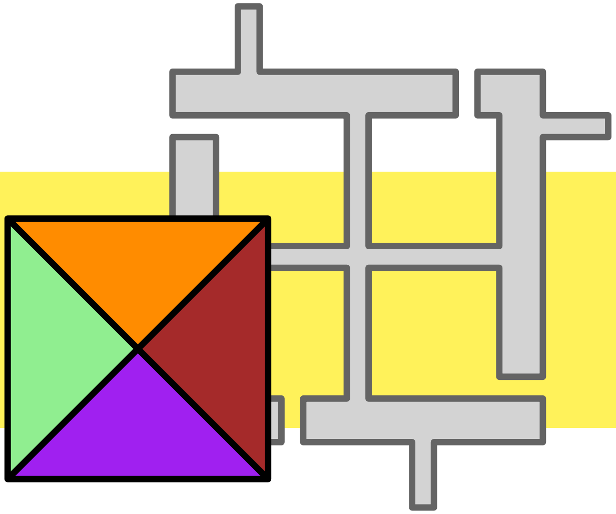
Clumsy Packings with Polyominoes

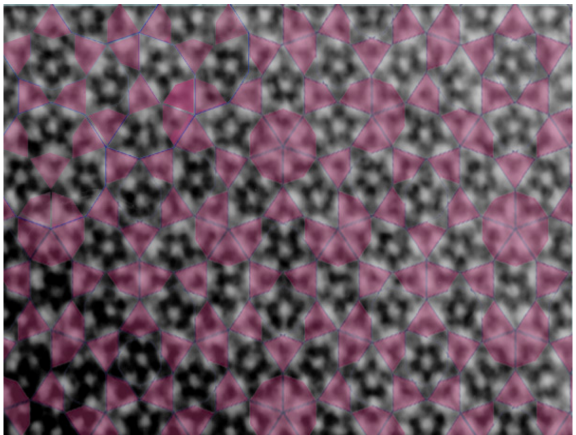
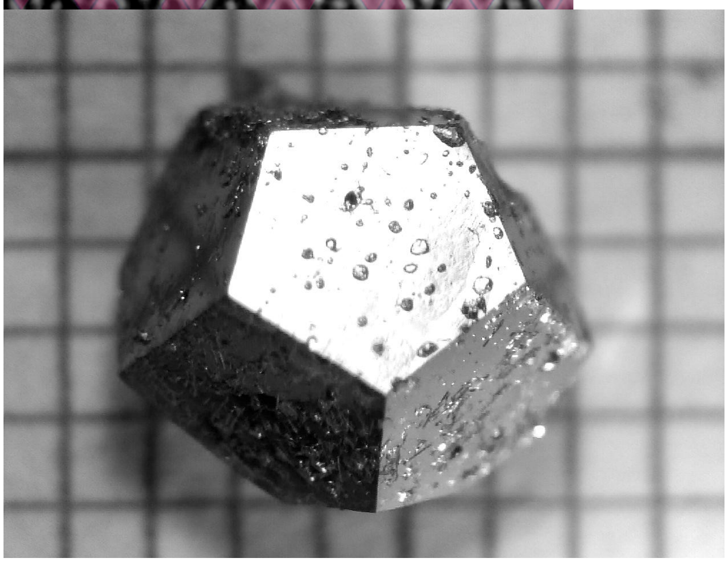
I. Introduction

- Definitions & Examples

II. Results

- Extremal Questions
- Aperiodic Clumsy Packings
- Undecidability




Almost-Clumsy Periodic Packings

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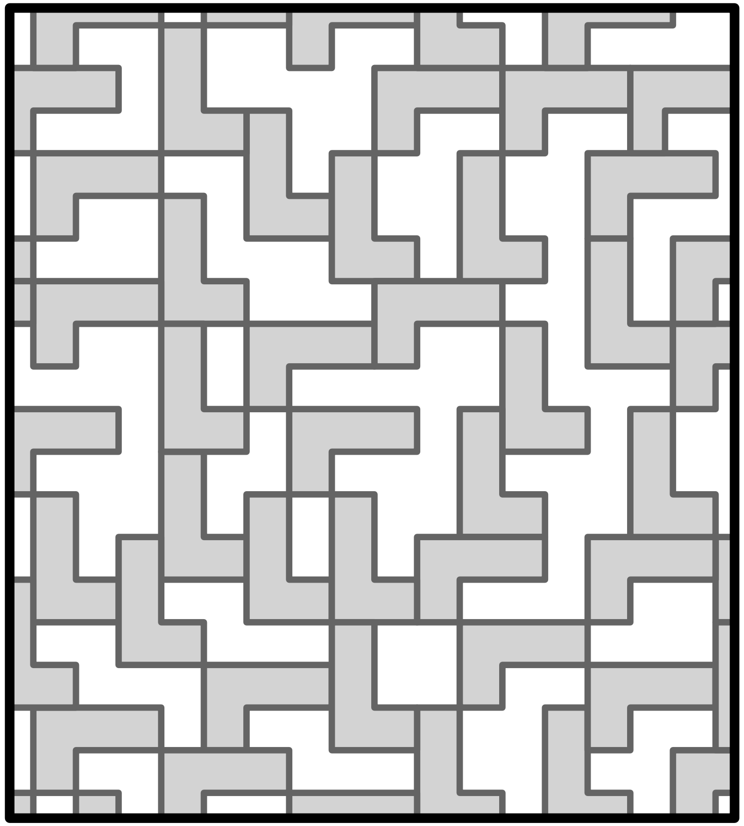

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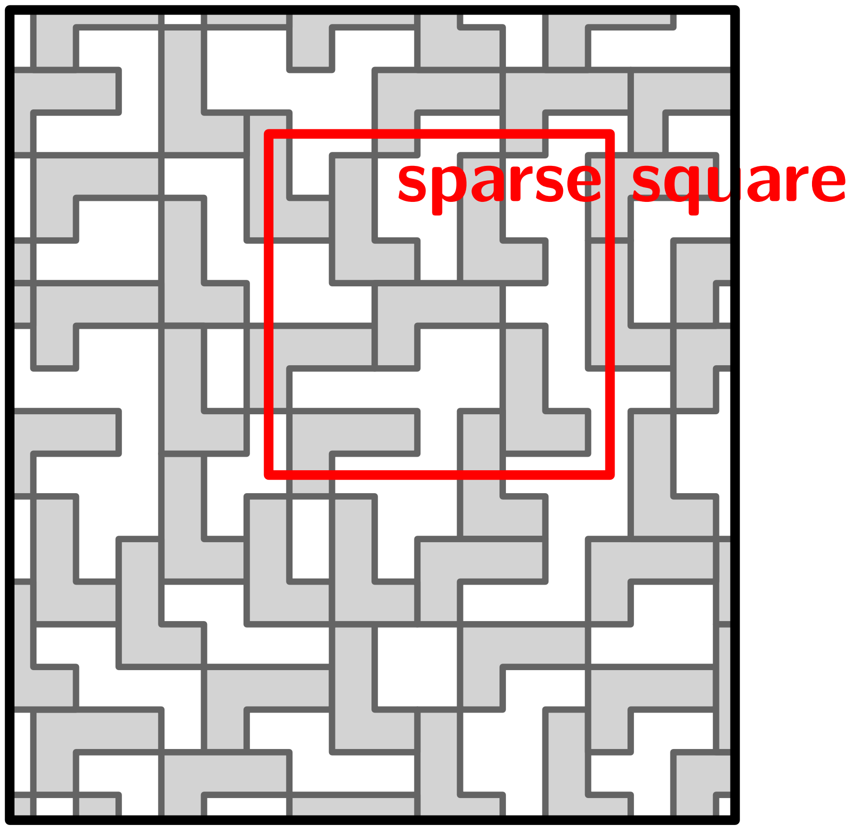

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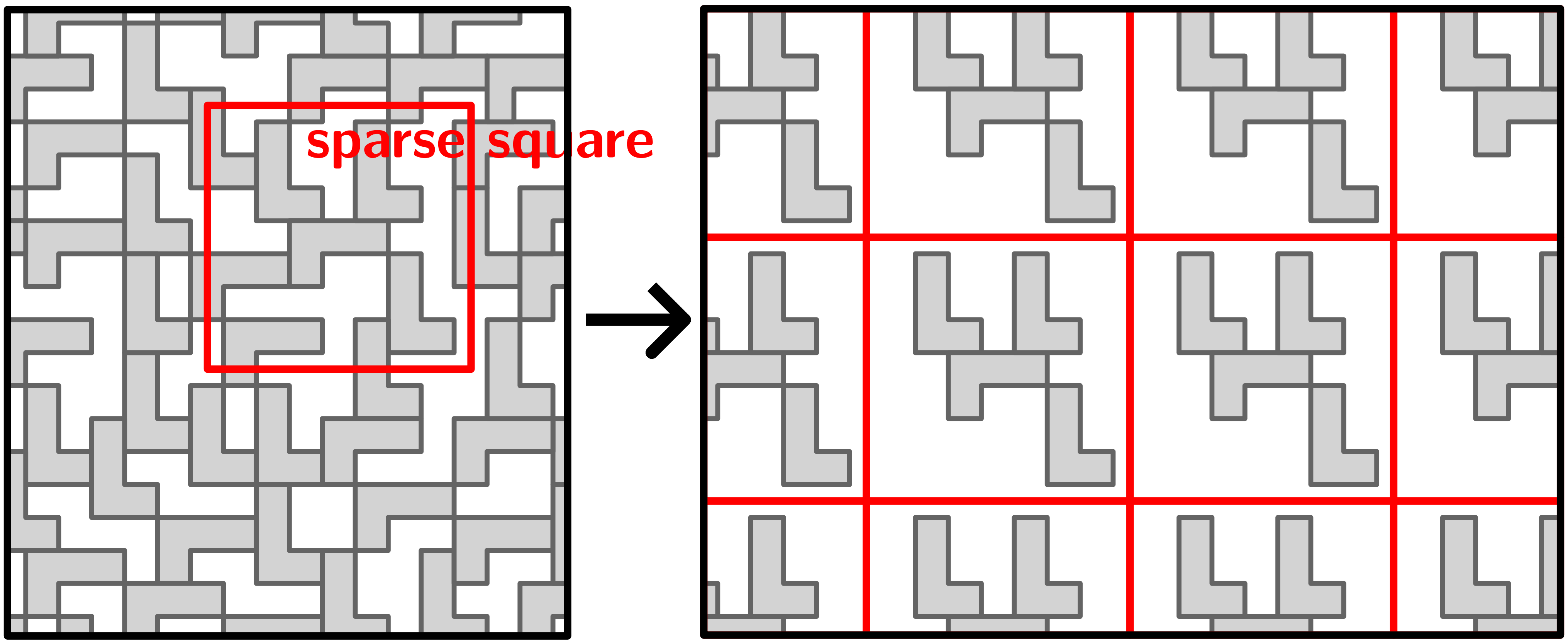

clumsy packing $\qquad$ ε-clumsy periodic packing

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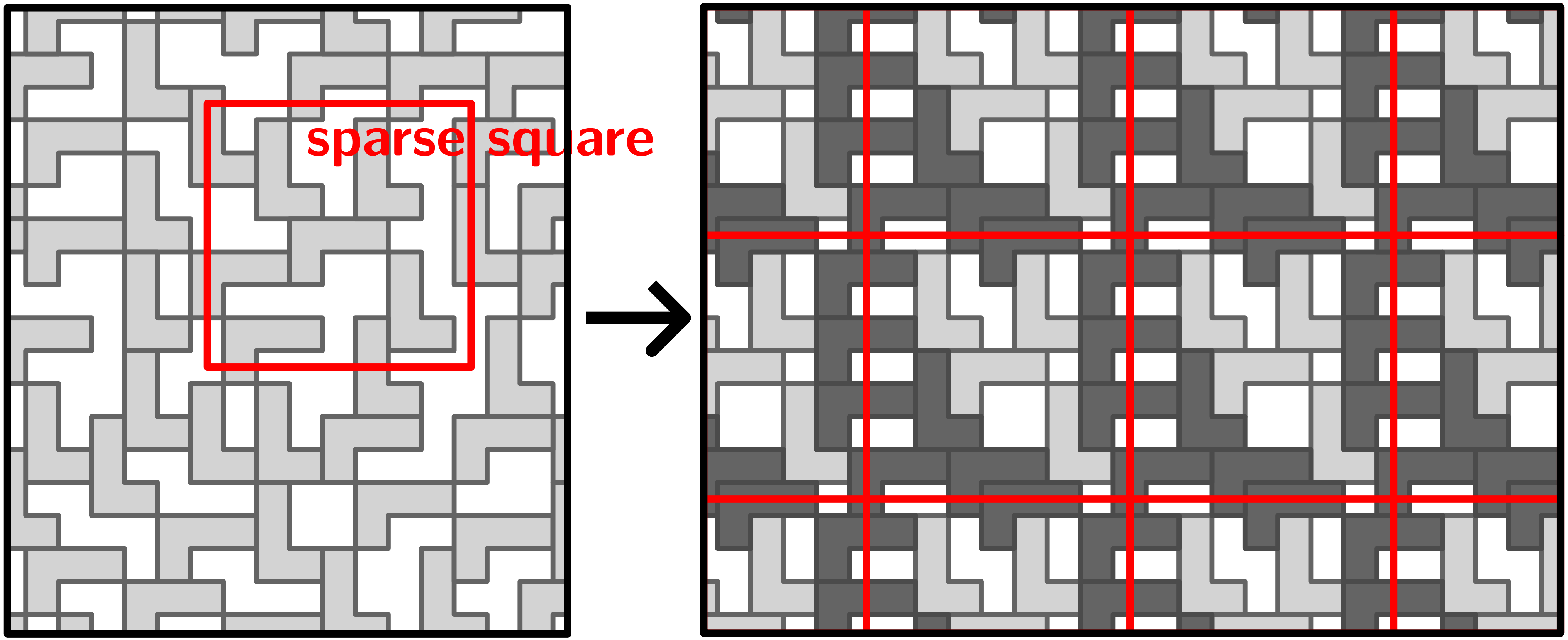

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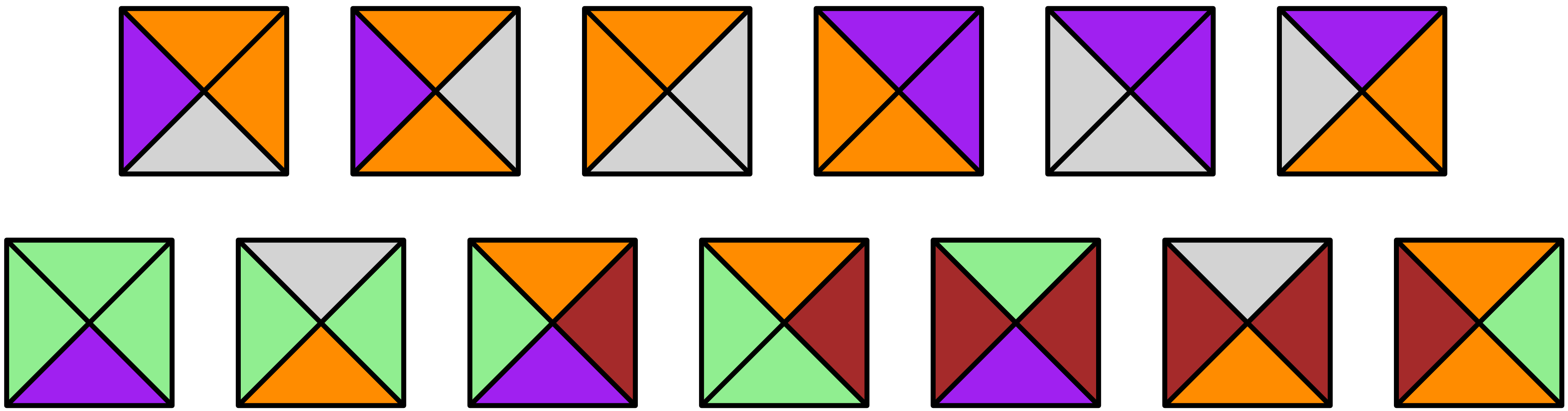
Wang Tiles



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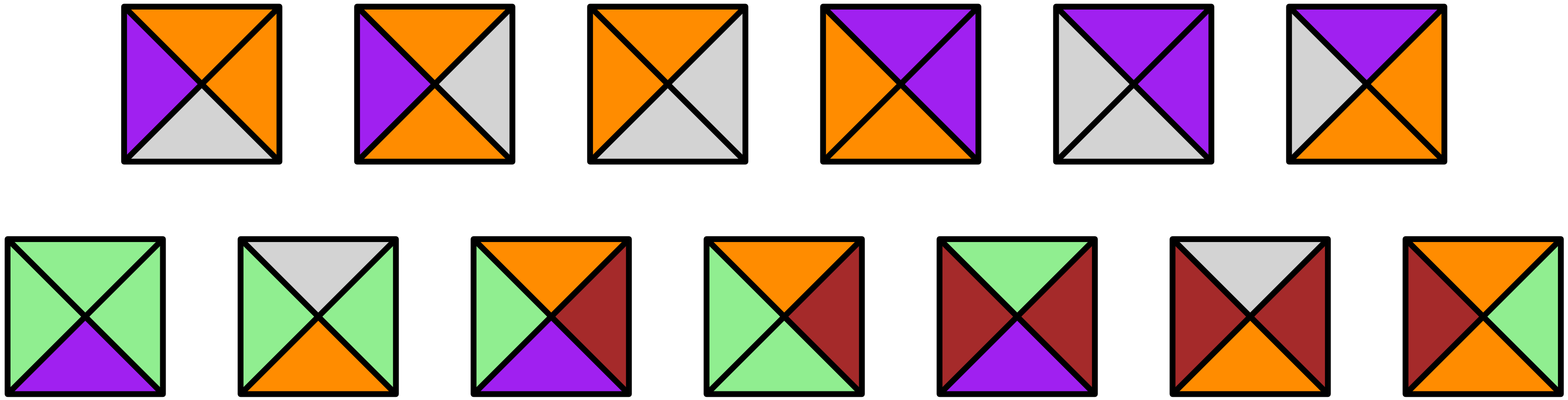

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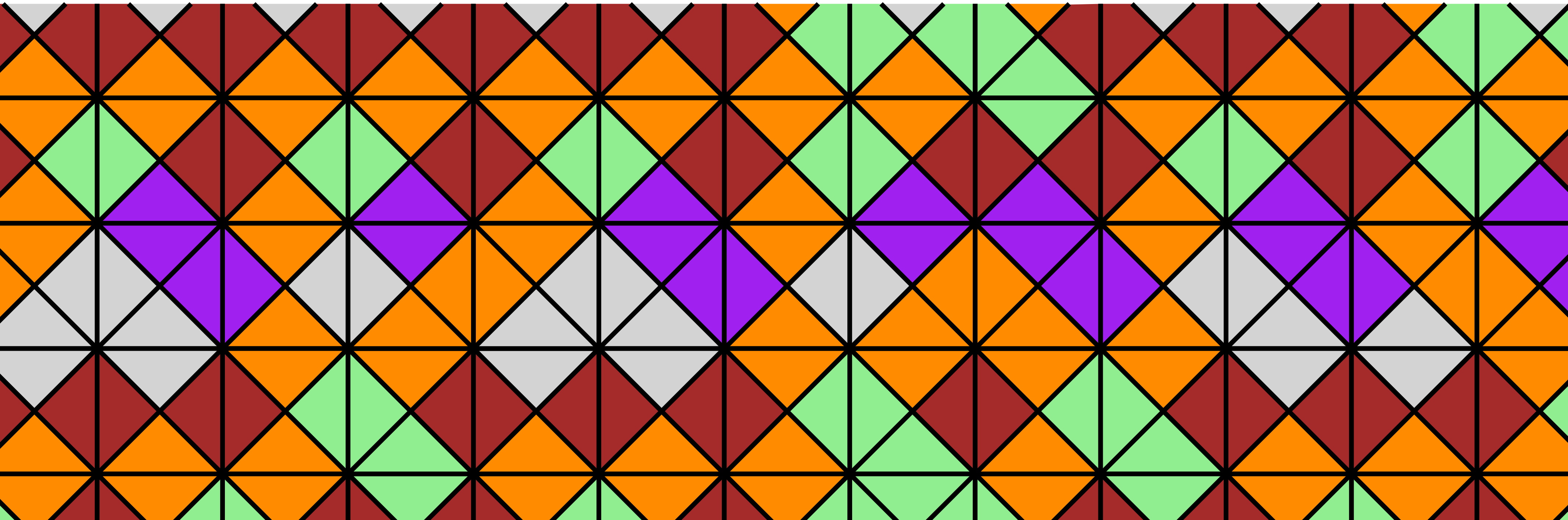
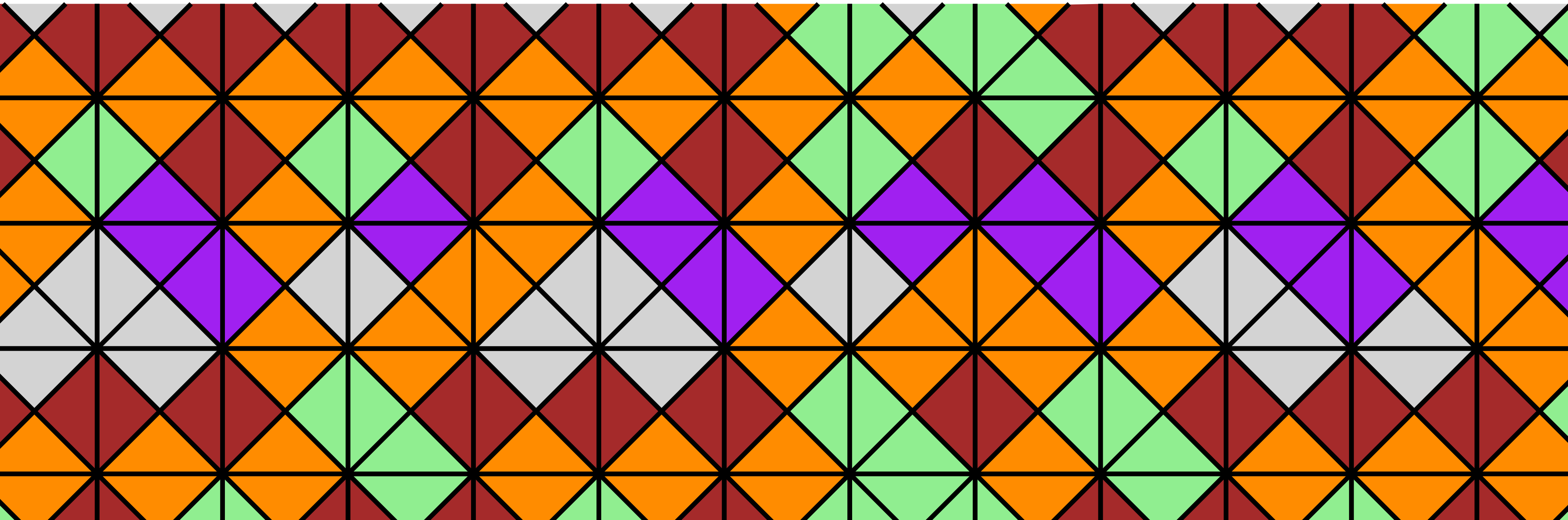


Wang Tiles

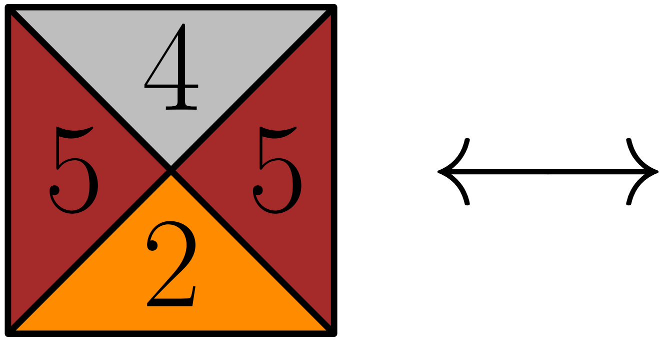
- **Thm.**(Culik 1996) There exists a set of 13 Wang tiles such that every Wang tiling is aperiodic.

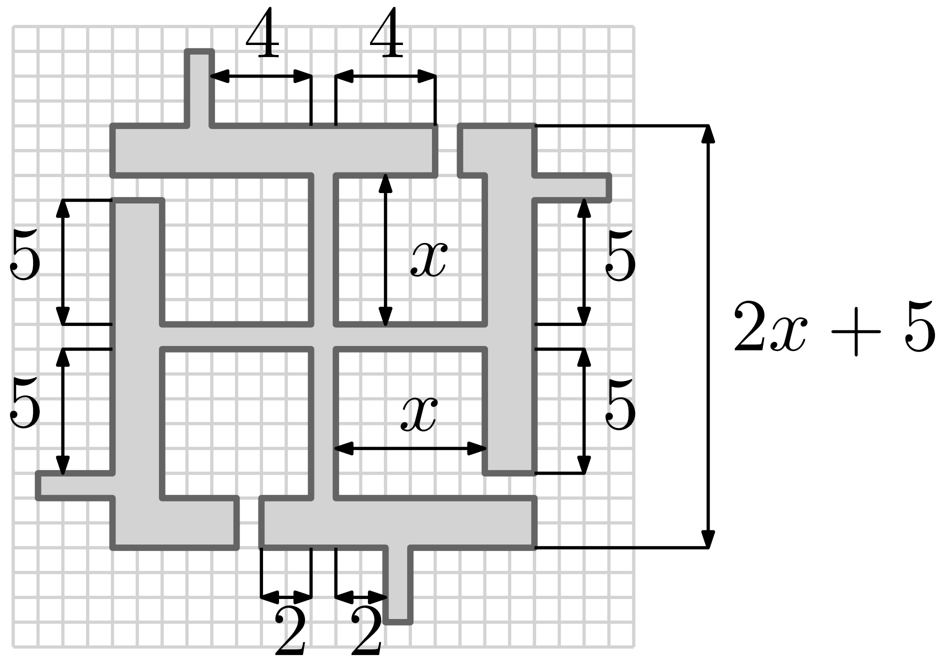
- **Thm.**(Berger 1966) It is undecidable whether a given set of Wang tiles tiles the plane.

- Wang Tiling



From Wang Tiles to Polyominoes





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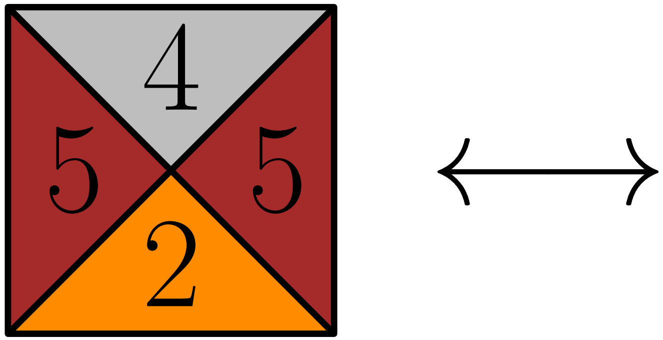
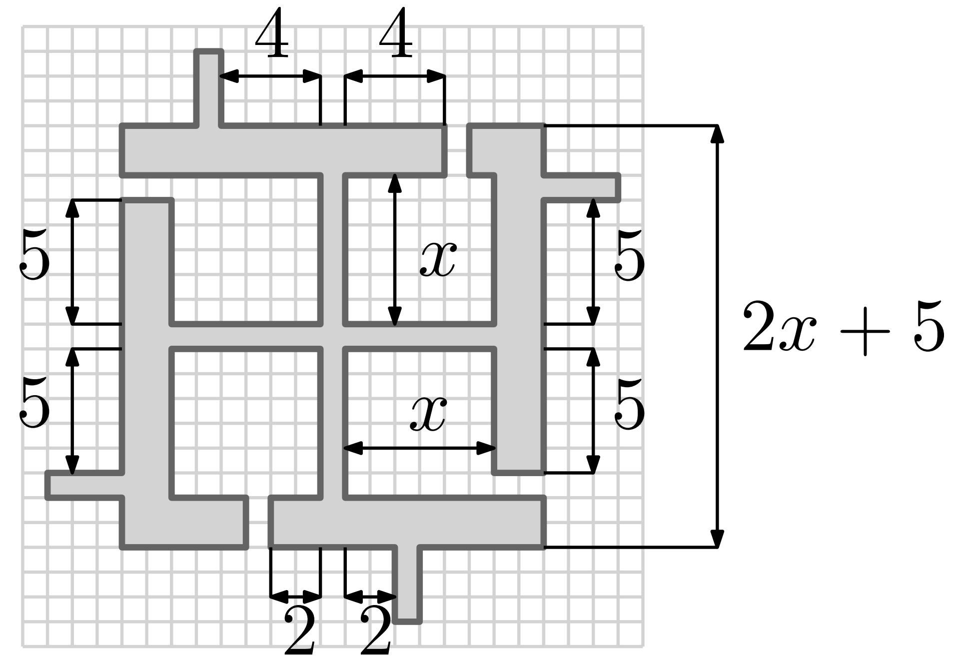
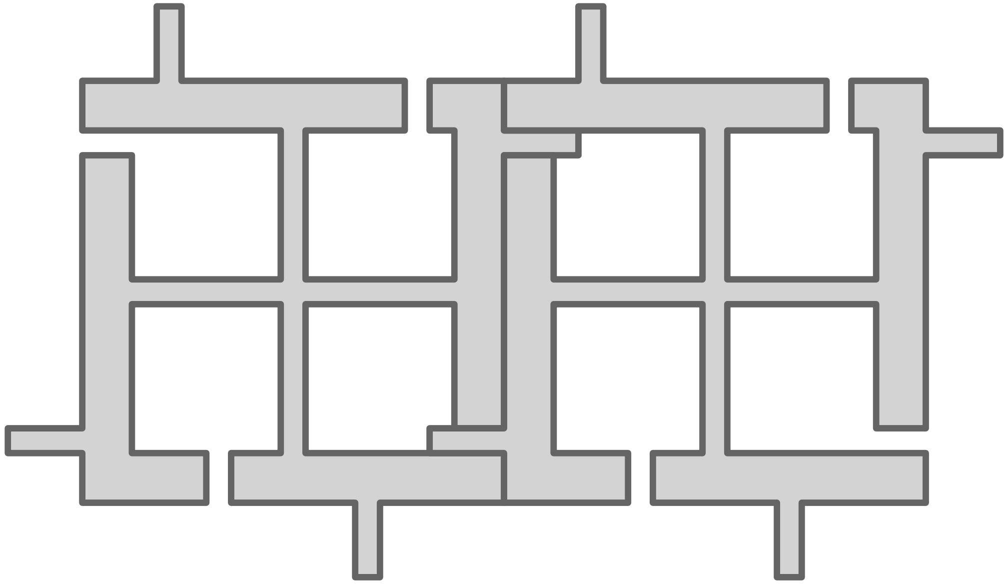
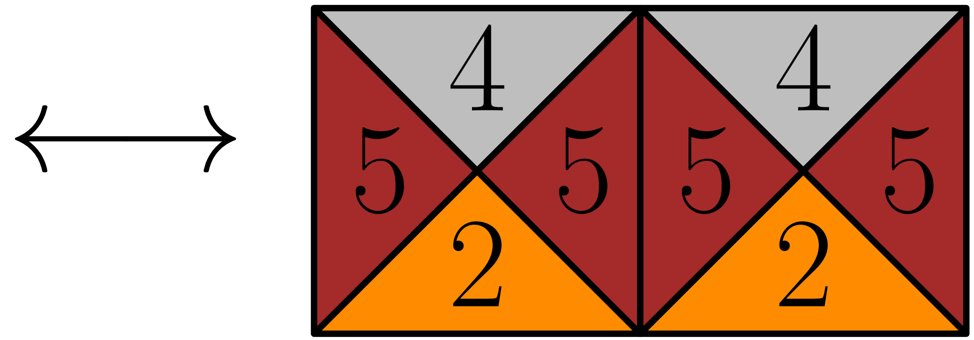


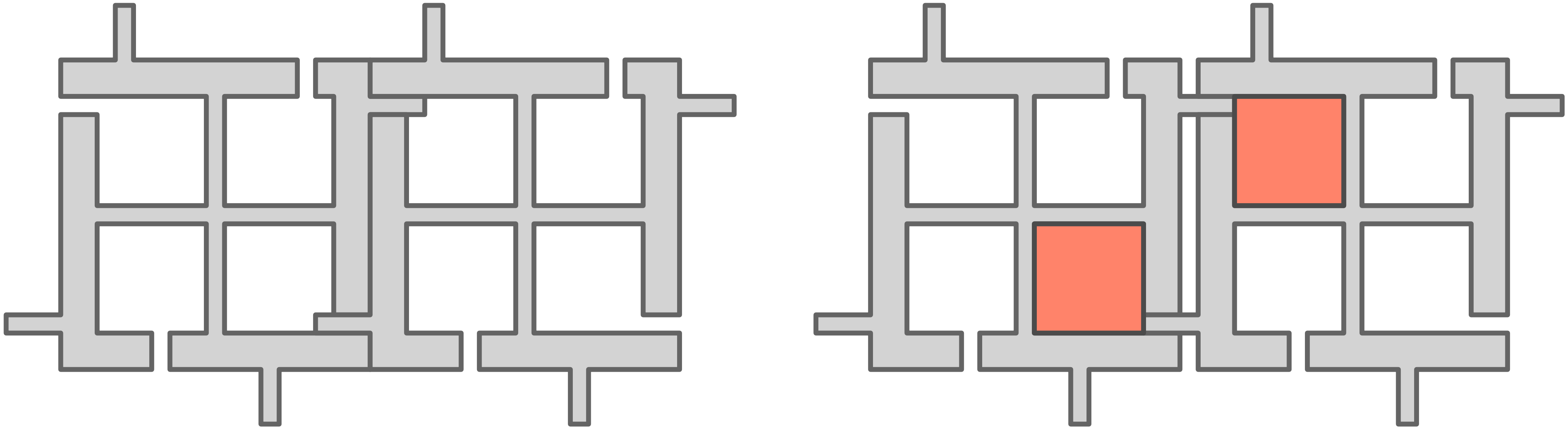






From Wang Tiles to Polyominoes

- Palette = Wang-polyominoes + bad x-by-x square



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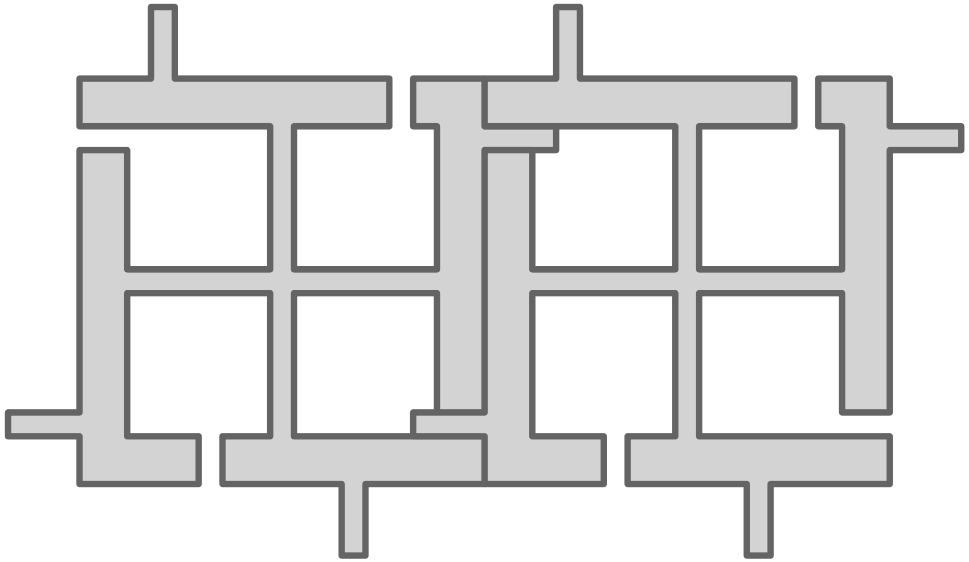

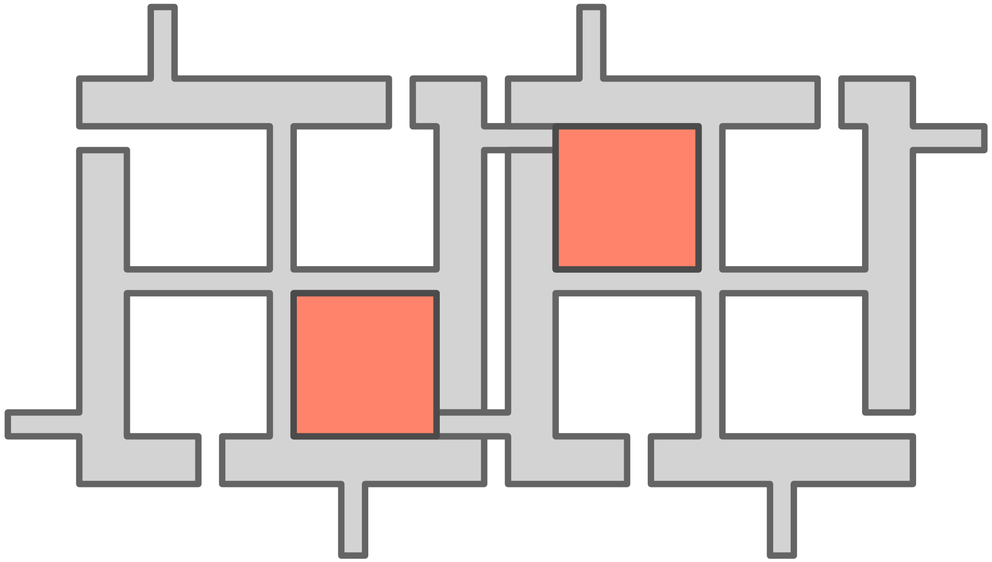

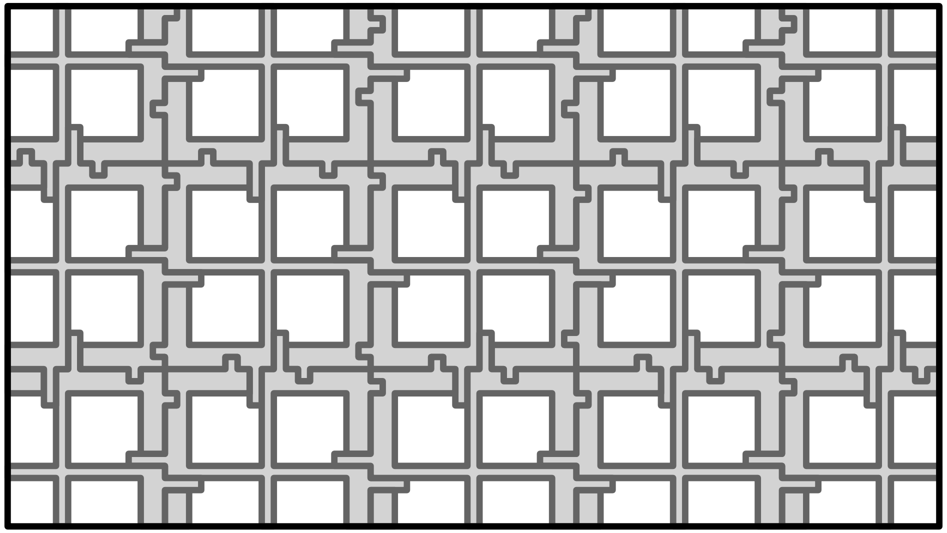

▷ Wang tiling exists

$$\Rightarrow \text{density}(P) = \frac{20x-29}{(2x+5)^2}.$$

▷ Wang tiling does not exist

⇒ "many" bad squares

$$\Rightarrow \text{density}(P) > \frac{20x-29}{(2x+5)^2}.$$

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Polyominoes

- **Theorem.** There exists a set of 14 polyominoes such that every clumsy packing is aperiodic.
- **Theorem.** For some $q \in \mathbb{Q}$ it is undecidable whether a given set of polyominoes has clumsiness at most q.

Thank you for your attention!

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Stefan Walzer stefan.walzer@student.kit.edu
Department of Mathematics, Karlsruhe Institute of Technology

Summary and Open Problems

- **Thm.** The clumsiest connected polyomino of size k has clumsiness $\approx 2/k$.

- **Open:** What is the clumsiest set of polyominoes each of size k?
  Open: What if we allow rotations?

- **Thm.** For every $\varepsilon > 0$ there exist a periodic packing P such that density$(P) \leq$ clumsiness $+\varepsilon$.

- **Thm.** Sometimes all clumsy packings are aperiodic.

- **Thm.** Computing clumsiness is undecidable for some $q \in \mathbb{Q}$.

- **Open:** What about other rational numbers q?