Exercises for Combinatorial and Computational Geometry

Series 4 — Duality and polytopes

hint 10.12.2024, deadline 14:00 17.12.2024

1. (Missing pieces of the proof that every V-polytope is also an H-polytope.)
   (a) Let $C \subseteq \mathbb{R}^d$ be a convex set. Prove that C^* is bounded if and only if 0 lies in the interior of C. **[2]**
   (b) Show that for every set $X \subset \mathbb{R}^d$, the second dual set $(X^*)^*$ is the closure of $\operatorname{conv}(X \cup \{0\})$. **[2]**
   (c) Let $P \subset \mathbb{R}^d$ be a V-polytope containing 0 in its interior. Show that P^* is the intersection of halfspaces dual to the vertices of P. **[1]**

2. A *convex body* is a bounded closed convex set in $\mathbb{R}^d$ whose interior contains 0. A convex body is *smooth* if for each point on its boundary there is exactly one tangent hyperplane. A convex body is *strictly convex* if its boundary contains no straight-line segment of positive length. Prove that a convex body K is strictly convex if and only if K^* is smooth. **[1]**

3. Let $v_1, \dots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. Let C be the convex hull of the rays $p_1, \dots, p_n$ that are determined by the vectors $v_1, \dots, v_n$ and start in the origin (that is, $p_i = \{x \in \mathbb{R}^n; (\exists \lambda \geq 0)\, x = \lambda v_i\}$).

   Prove that there is a ray in C that forms an acute angle with every ray p_i. **[3]**

4. Consider n line segments in the plane such that each of them is contained in a line passing through the origin, but none of these line segments contains the origin. Show that if every triple of the line segments can be intersected by a common line, then all n line segments can be intersected by a common line. (By intersecting we mean that the line segment and the line have at least one point in common. In particular, a line containing a line segment intersects this line segment.) **[3]**

5. Prove that every polytope $P \subset \mathbb{R}^d$ is an orthogonal projection of some k-dimensional regular simplex in $\mathbb{R}^n$ for suitable k, n. (An *orthogonal projection* is a mapping π from the space $\mathbb{R}^n$ to a subspace $M \cong \mathbb{R}^d$ embedded in $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ the vector $\pi(x) - x$ is orthogonal to M. A simplex is *regular* if all its edges have the same length.) **[4+hint]**

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web: `https://kam.mff.cuni.cz/~kvgweb/kvg/eng.html`