Chapter 6

Halving segments

Definition 6.1. Given a set P of n points in the plane in general position, a segment s connecting two points of P is called a **halving segment** or a **halving edge** if each open halfplane determined by s contains $\lfloor (n-2)/2 \rfloor$ or $\lceil (n-2)/2 \rceil$ points of P. That is, the number of points of P on the left side of s is the same as the number of points of P on the right side of s if n is even, or the two numbers differ by 1 if n is odd. The line extending a halving segment is called a **halving line**.

Observe that in a set of n points in convex position in the plane, there are exactly $n/2$ halving segments if n is even, and n halving segments if n is odd.

6.1 Upper bounds

What is the maximum possible number of halving segments of a set of n points in the plane? Lovász [44] obtained the upper bound $O(n^{3/2})$, which was later improved by Pach, Steiger and Szemerédi [54] to $O(n^{3/2}/\log^* n)$. The best known bound is $O(n^{4/3})$, first proved by Dey [19]. First we present Lovász' approach.

Theorem 6.2 (Lovász, 1971 [44])**.** *For n even, the maximum number of halving segments in a set of n points in the plane in general position is $O(n^{3/2})$.*

Proof. It is an easy exercise to show that the geometric graph G formed by halving segments has the following property: for every vertex v and every pair of halving segments s_1, s_2 (edges of G) incident to v, the cone opposite to the convex cone determined by s_1 and s_2 contains another halving segment incident with v. By the same property, the cone contains, in fact, exactly one halving segment incident with v. Also, every vertex has at least one halving segment incident to it. If we assume that no two vertices of G lie on a vertical line, it follows that the degree of every vertex is odd and the number of neighbors of v that are to the left of v differs from the number of right neighbors by 1.

The crucial observation is that every vertical line intersects at most $n/2$ halving segments. To show this, start with a vertical line p that crosses k halving segments. Assume without loss of generality that p has at most $n/2$ vertices on its left. Start translating p to the left. The number of halving segments intersected by p changes only when p passes through a vertex, and then it changes by exactly 1. After at most $n/2$ such changes all the vertices will be to the right of p, so p will not intersect any halving segment. Therefore, $k \leq n/2$.

To finish the proof, draw vertical lines $p_1, p_2, \ldots, p_{\lceil\sqrt{n}\rceil}$ so that in every region between p_i and p_{i+1}, to the left of p_1, and to the right of $p_{\lceil\sqrt{n}\rceil}$, there are at most $\sqrt{n}$ vertices. The number of halving segments crossing at least one of the lines p_i is $O(n\sqrt{n})$, since every vertical line intersects $O(n)$ halving segments. The number of halving segments that are disjoint with all the lines p_i is $O(n\sqrt{n})$, since each of the $\lceil\sqrt{n}\,\rceil + 1$ regions contains only $O(n)$ pairs of vertices. □

The following remarkable identity combined with the crossing lemma gives an improvement on the Lovász' bound.

Theorem 6.3 (Andrzejak et al., 1998 [7]). *Let n be an even positive integer. Let G be the geometric graph determined by the halving segments of n points in the plane in general position. Let k be the number of crossings in G. Then we have*

$$k + \sum_{v \in V(G)} \binom{(d(v)+1)/2}{2} = \binom{n/2}{2}.$$

Idea of the proof. Start with n points in convex position and move them continuously one by one to the vertices of G. In convex position there are $n/2$ halving segments, every two of them cross, and each of the n points is incident with exactly one halving segment, so the equality holds. During the continuous motion of the vertices, the elementary changes to the graph of halving segments do not affect the validity of the identity. □

Theorem 6.4. *For n even, the maximum number of halving segments in a set of n points in general position in the plane is $O(n^{4/3})$.*

Proof. Let G be the graph of halving segments. Theorem 6.3 implies that $\mathrm{cr}(G) = O(n^2)$. However, by the crossing lemma either $e(G) = O(n)$ or $cr(G) = \Omega(e^3/n^2)$. In any case, $e(G) = O(n^{4/3})$. □

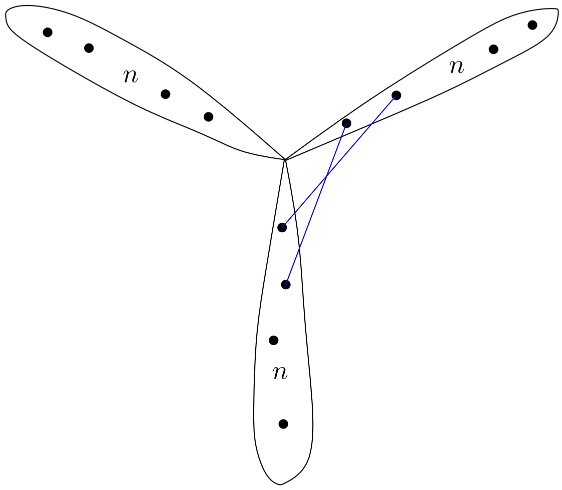

Figure 6.1: Inductive construction for halving segments. Two of the "new" $3n/2$ halving segments are drawn.

6.2 Lower bounds

Theorem 6.5 (Strauss, 1973 [23]). *For n even, there are sets of n points in general position in the plane with $\Omega(n \log n)$ halving segments.*

Proof. The construction is done by induction. Suppose we have a construction with n points and $h(n)$ halving segments. Then we can build a configuration with $3n$ points and $h(3n) \geq 3h(n) + cn$ halving segments, for a certain constant $c > 0$, as follows. We squash the configuration of n points to make it look almost like a segment, we take three copies of it and arrange them in three directions separated by 120° as in Figure 6.1. We have $h(3n) \geq 3h(n) + 3n/2$, which gives $h(n) = \Omega(n \log n)$. when h is nondecreasing. It is a simple exercise to show that from a configuration of n points with s halving segments one can create a configuration of $n + 2$ points with s halving segments. □

The lower bound has been significantly improved: Tóth [67] constructed a set of n points with $ne^{\Omega(\sqrt{\log n})}$ halving segments. Nivasch [48] simplified the construction and improved the lower bound to $\Omega(ne^{\sqrt{\ln 4}\sqrt{\ln n}}/\ln n)$. (We use "log" for the binary logarithm and "ln" for the natural logarithm.)

Theorem 6.6 (Nivasch, 2008 [48]). *For n even, there are sets of n points in general position in the plane with $\Omega(ne^{\sqrt{\ln 4}\sqrt{\ln n}}/\ln n)$ halving segments.*

Instead of a set of points, we construct a *dual* set of lines. We use the following notion of duality.

Definition 6.7. Given a point $p = (c, d) \in \mathbb{R}^2$, the **dual** (or the **dual line**) of p is the (non-vertical) line $p^* = \{(x, y) \in \mathbb{R}^2; y = cx - d\}$. Given a non-vertical line $\ell = \{(x, y) \in \mathbb{R}^2; y = ax - b\}$, the **dual** (or the **dual point**) of ℓ is the point $\ell^* = (a, b) \in \mathbb{R}^2$.

The following observation is left as an exercise.

Observation 6.8. *Suppose that p is a point in $\mathbb{R}^2$ and ℓ is a non-vertical line in $\mathbb{R}^2$. Then*

a) *$(p^*)^* = p$ and $(\ell^*)^* = \ell$,*

b) *$p \in \ell$ if and only if $\ell^* \in p^*$,*

c) *p lies above ℓ if and only if p^* lies below ℓ^*.*

Definition 6.9. A finite set L of lines in the plane forms an **arrangement of lines** $\mathcal{A}(L)$, which is the decomposition of the plane into **vertices**, **edges** and **cells**, where the vertices are the intersection points of the lines, the edges are the open segments or rays of the lines that remain after removing the vertices, and the cells are the 2-dimensional open regions that are the connected components of $\mathbb{R}^2 \setminus (\bigcup L)$. The vertices, edges and cells are also called 0-, 1- and 2-dimensional **faces** of the arrangement, respectively.

The **level** of a point p with respect to a set of lines L (or with respect to an arrangement $\mathcal{A}(L)$) is the number of lines of L that lie strictly below p. The **level** of a face of $\mathcal{A}(L)$ is the level of an arbitrary point of the face with respect to $\mathcal{A}(L)$.

The following observation follows from Observation 6.8.

Observation 6.10. *Let n be even and let P be a set of n points in the plane in general position. Let L be the set of n dual lines of the points of P, and suppose that L is in general position, that is, no two lines of L are parallel, no three lines of L pass through the same point. Moreover, suppose that no line of L is vertical. Then the dual of a halving line of P is a vertex of $\mathcal{A}(L)$ of level $n/2 - 1$.*

We will call the vertices of $\mathcal{A}(L)$ of level $n/2 - 1$ the **middle-level vertices**. Similarly, the cells of $\mathcal{A}(L)$ of level $n/2$ will be called the **middle-level cells**. By Observation 6.10, constructing a point set with many halving lines is equivalent to constructing an arrangement of lines with many middle-level vertices.

Proof of Theorem 6.6

First we describe the construction, then we verify its correctness, and finally we count the middle-level vertices.

The construction

We construct an infinite sequence $L_0, L_1, L_2, \ldots$ of sets of non-vertical lines in the plane in general position. Every line in every L_m, $m \geq 0$, is of one of two types: **plain** or **bold**. For every L_m, we construct a set V_m of middle-level vertices of $\mathcal{A}(L_m)$ such that each of them lies in the intersection of a plain line and a bold line. The set V_m does not necessarily contain all the middle-level vertices with this property. The construction depends on free parameters $a_0, a_1, a_2, \ldots$, which we choose as $a_0 = 0$ and $a_m = 2^m$ for $m \geq 1$.

The base case, L_0, consists of a plain line ℓ_0, a bold line b_0, and a vertex v_0 in their intersection. We set $V_0 = \{v_0\}$.

Now we describe the inductive step, which is the heart of the construction. Let $m \geq 0$ and suppose that L_m and V_m have been constructed. We construct L_{m+1} and V_{m+1} as follows.

Each plain line $\ell \in L_m$ is replaced by a *bundle* of a_{m+1} plain lines parallel and close to ℓ separated by a very small distance $\varepsilon_m > 0$. The new lines are then slightly perturbed into general position, but only so little that within a square containing V_m, their displacement is almost imperceptible and they still appear *almost parallel*. Each bold line $b \in L_m$ is replaced by a bundle of $a_{m+1} + 1$ plain lines parallel and close to b separated by a very small distance $\delta_m > 0$ that is much smaller than ε_m. Again, the new lines are then slightly perturbed into general position. We will call this step a *uniform replacement*.

For every vertex $v \in V_m$, the uniform replacement creates an $a_{m+1} \times (a_{m+1} + 1)$ grid G_v in place of v; see Figure 6.2. We then draw a new bold line b'_v along the diagonal of the grid, so that its crossings with the lines of the two bundles alternate. We add these $2a_{m+1} + 1$ vertices of L_{m+1} to the set V_{m+1}. We assume that δ_m is so small compared to ε_m, that b'_v is very close to the original bold line in L_m that contained v.

The correctness

We need to verify that all the vertices in V_{m+1} are, indeed, middle-level vertices of $\mathcal{A}(L_{m+1})$. We will show a stronger property, which is needed to show this by induction. We say that a point v is **strongly balanced** in a subset L of L_m if the number of plain lines in L above v is equal to the number of plain lines in L below v, and the number of bold lines in L above v is equal to the number of bold lines in L below v.

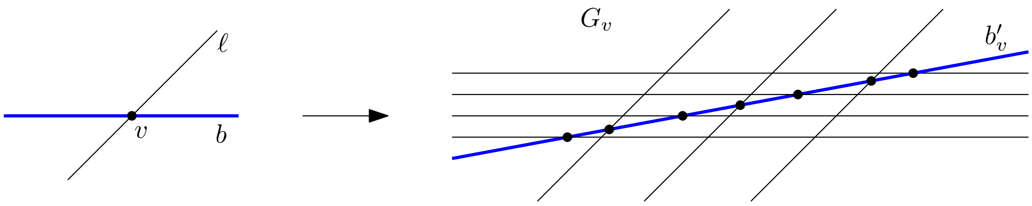

Figure 6.2: A uniform replacement creating the grid G_v from v, and a new bold line b'_v. The vertex v of V_m is replaced by $2a_m + 1$ vertices in V_{m+1}. Here $a_m = 3$.

Lemma 6.11. *For every $m \geq 0$, all vertices in V_m are strongly balanced in L_m.*

Proof. We proceed by induction on m. For $m = 0$, the vertex v_0 in L_0 is strongly balanced in L_0, since there are no lines of L_0 above or below it. For the induction step, let $m \geq 0$ and suppose that the lemma is true for all vertices of V_m. We will prove it for the vertices of V_{m+1}.

Each bold line b in L_m contains $2a_m + 1$ vertices of V_m. Moreover, the slopes of the plain lines passing through these vertices alternate between larger and smaller than the slope of b, since these plain lines form a grid in the construction. Let $v \in V_m$. Let b be the bold line and ℓ the plain line containing v. Let b'_v be the bold line of L_{m+1} through the grid G_v. Let $w \in V_{m+1} \cap b'_v$. Let $v_1, v_2, \ldots, v_{2a_m}$ be the vertices of $V_m \cap b$ other than v, and let $b'_1, b'_2, \ldots, b'_{2a_m}$ be the corresponding bold lines in L_{m+1}.

Partition L_{m+1} into three sets, S_1, S_2 and S_3, as follows. Let S_1 be the set of all lines of L_{m+1} created from lines other than ℓ or b (the set S_1 contains both plain and bold lines). By induction, v is strongly balanced in L_m. This implies that after the uniform replacement, w is strongly balanced in S_1.

Let S_2 be the set of lines of the grid G_v and the line b'_v; see Figure 6.2. The location of b'_v along the diagonal of the grid implies that w is strongly balanced in S_2.

Let $S_3 = \{b'_1, b'_2, \ldots, b'_{2a_m}\}$. These lines, together with b'_v, are created from b, and also satisfy the property that their slopes alternate between larger and smaller than the slope of b. If there are an even number of the points $v_1, v_2, \ldots, v_{2a_m}$ to the right of v (and even number to the left of v), half of the corresponding lines b'_i is above v (and thus above w) and the other half below v (and thus below w). If there are an odd number of points $v_1, v_2, \ldots, v_{2a_m}$ to the right of v (and odd number to the left of v), the two lines b'_i and b'_j corresponding to the points v_i and v_j closest to v from left and right, respectively, have both larger or both smaller slope than b. But this means that one of the lines b'_i, b'_j is above v (and w) and the other below v (and w). The rest follows from the previous even case. This shows that w is strongly balanced in S_3. Altogether, w is strongly balanced in $S_1 \cup S_2 \cup S_3 = L_{m+1}$. $\square$

Computations

For $m \geq 0$, let $n_m = |L_m|$ and $f_m = |V_m|$. Recall that $a_0 = 0$ and $a_m = 2^m$ for $m \geq 1$. From the construction of L_0 we have $n_0 = 2$ and $f_0 = 1$. By the construction of V_{m+1}, we have

$$f_{m+1} = (2a_{m+1} + 1) \cdot f_m.$$

Now we count the number of lines in L_{m+1}. For every $i \geq 1$, the number of bold lines in L_i is equal to the number of vertices in V_{i-1}, which is equal to f_{i-1}. The number of plain lines in L_i is thus $n_i - f_{i-1}$. By the uniform replacement, it follows that the number of plain lines in L_{m+1} is $a_{m+1} \cdot (n_m - f_{m-1}) + (a_{m+1} + 1) \cdot f_{m-1} = a_{m+1} n_m + f_{m-1}$. The number of bold lines in L_{m+1} is f_m. Together, the number of lines in L_{m+1} is

$$n_{m+1} = a_{m+1} n_m + f_m + f_{m-1}.$$

Now by a straightforward induction, we have

$$f_m = f_0 \cdot (2a_1 + 1)(2a_2 + 1) \cdots (2a_m + 1) = (2^2 + 1)(2^3 + 1) \cdots (2^{m+1} + 1).$$

It is an easy exercise to show that $f_m = \Theta\big(2^{(m^2+3m)/2}\big)$. Plugging this into the recursion for n_m, we get

$$n_m = 2^m \cdot n_{m-1} + \Theta(f_{m-1}) = 2^m \cdot n_{m-1} + \Theta\big(2^{(m^2+m)/2}\big).$$

Let $n'_0, n'_1, n'_2, \ldots$ be a sequence satisfying the recursion $n'_m = 2^m \cdot n'_{m-1} + k \cdot 2^{(m^2+m)/2}$, where k is a constant. It is a straightforward exercise to verify that $n'_m = 2^{(m^2+m)/2} \cdot (n'_0 + km)$. Since the sequence n_m is "sandwiched" between two sequences of the type n'_m just with a different constant k, we conclude that $n_m = \Theta\big(m \cdot 2^{(m^2+m)/2}\big)$. This further implies that

$$\begin{aligned} \log n_m &= \log m + \frac{m^2 + m}{2} + \Theta(1) \\ \Rightarrow\ 2\log n_m &= m^2 + m + 2\log m + \Theta(1) = \big(m + \Theta(1)\big)^2 \\ \Rightarrow\ m &= \sqrt{2\log n_m} - \Theta(1) \end{aligned}$$

and also that

$$\frac{f_m}{n_m} = \Theta(2^m/m).$$

Combining the last two expressions we get

$$\begin{aligned} f_m &= \Theta\left(n_m \cdot 2^{\sqrt{2\log n_m} - \Theta(1)} / \left(\sqrt{2\log n_m} - \Theta(1)\right)\right) \\ &= \Theta\left(n_m \cdot 2^{\sqrt{2\log n_m}} / \sqrt{\log n_m}\right) \\ &= \Theta\left(n_m \cdot e^{\ln 2\sqrt{2}\sqrt{\ln n_m}/\sqrt{\ln 2}} / \sqrt{\ln n_m}\right) \\ &= \Theta\left(n_m \cdot e^{\sqrt{\ln 4}\sqrt{\ln n_m}} / \sqrt{\ln n_m}\right). \end{aligned}$$

We have finished the proof of Theorem 6.6 for $n = n_m$. To prove it for all even n, we need to "fill the gaps" between consecutive members of the sequence n_m [67]. For this, we need to observe that if a set of n points in the plane in general position has s halving segments, then we can add two points so that the resulting set still has at least s halving segments (thus the maximum number of halving segments is a nondecreasing function for even n). But this is not enough, since the sequence n_m grows very fast. So we use a second observation, stating that if a set of n points in the plane has s halving segments and a is a positive integer, then there is a set of an points in the plane with at least as halving segments. This will be sufficient to interpolate the lower bound on the number of halving segments for even values of n and finish the proof of the theorem. This, including the two observations, is left as an exercise.

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