Exercises for Combinatorial and Computational Geometry

Series 2 — Helly-type theorems and the ham sandwich theorem

hints 3.11.2020, deadline 9.11.2020

1. Let $C_1, \ldots, C_n$ be a collection of at least three convex sets in the plane and let K be the line segment $[0,1] \times \{0\}$. Show that if the intersection of every triple of sets from $C_1, \ldots, C_n$ contains a translated copy of K, then also the intersection of all the sets $C_1, \ldots, C_n$ contains a translated copy of K. **[2]**

2. Let $M \subset \mathbb{R}^2$ be a closed polygon with area $S(M)$, not necessarily convex. Prove that there is a point $x \in \mathbb{R}^2$ such that every line that contains x divides M into two parts, each with area at least $S(M)/3$. **[2]**

3. A collection $\mathcal{C} = \{C_1, \ldots, C_n\}$ of convex sets in the plane has (p,q)-*property* if $n \geq p$ and every p-tuple of sets from $\mathcal{C}$ contains a q-tuple with a nonempty intersection. The *piercing number* $s(\mathcal{C})$ of the collection $\mathcal{C}$ is the minimum size of a set $X \subset \mathbb{R}^2$ such that every $C_i \in \mathcal{C}$ contains at least one point from X.

   (a) Prove that every finite collection $\mathcal{C}$ of closed axis-parallel rectangles with the $(2,2)$-property satisfies $s(\mathcal{C}) = 1$. **[2]**

   (b) Prove that every finite collection $\mathcal{C}$ of closed axis-parallel rectangles with the $(4,3)$-property satisfies $s(\mathcal{C}) \leq 2$. **[3]**

4. For $n \geq 4$, let $C_1, \ldots, C_n$ be a collection of convex sets in the plane. Show that if the intersection of every 4-tuple of sets from $C_1, \ldots, C_n$ contains a ray, then the intersection of all the sets $C_1, \ldots, C_n$ contains a ray. **[4, hint]**

5. The ham sandwich theorem says that for every collection of disjoint finite sets $A_1, \ldots, A_d \subset \mathbb{R}^d$ there is a hyperplane h such that every open halfspace determined by h contains at most $\left\lfloor \frac{|A_i|}{2} \right\rfloor$ points of every A_i.

   Let $A_1, A_2, \ldots, A_d$ be disjoint sets in $\mathbb{R}^d$ such that every A_i contains n points and the points in $\bigcup_{i=1}^d A_i$ are in general position (that is, no hyperplane contains more than d points from this union). Show that the points from $\bigcup_{i=1}^d A_i$ can be partitioned into n *rainbow* d-tuples (that is, sets $\{x_1, x_2, \ldots, x_d\}$ with $x_i \in A_i$) such that their convex hulls are disjoint.

   If you wish to use a different version of the ham sandwich theorem than the one stated here, prove it first. **[2]**