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NOTE

ON THE STRUCTURE OF GRAPHS WITH BOUNDED CLIQUE NUMBER

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In this note, a structural result for maximal K_r-free graphs is proven, which provides a simple proof of the Andrásfai–Erdős–Sós Theorem, saying that every K_r-free graph with minimum degree $\delta > (1 - \frac{1}{r-\frac{4}{3}})n$ is $(r-1)$-colourable.

Turán's famous Theorem [3] from 1941 states that every K_r-free graph on n vertices has at most as many edges as the complete $(r-1)$-partite graph on n vertices, all of whose partite sets have cardinality differing by at most 1. This so-called Turán graph is uniquely determined (up to isomorphism) by this property and it is the only extremal graph. This implies that a K_r-free graph has minimum degree $\delta \le (r-2)n/(r-1)$. The Turán graph itself, appearing also here as an extremal graph, is $(r-1)$-colourable. Since the clique number is a lower bound for the chromatic number of a graph, the $(r-1)$-colourable graphs can be considered as those with a particularly simple structure among the graphs with clique number $r-1$.

In 1974, Andrásfai, Erdős and Sós [1] proved a lower bound on the minimum degree which ensures that a K_r-free graph is $(r-1)$-colourable:

Theorem 1 (Andrásfai, Erdős and Sós). *Let G be a K_r-free graph. If $\delta(G) > (1 - \frac{1}{r-\frac{4}{3}})n$ then G is $(r-1)$-colourable.*

The original proof in [1] is fairly lengthy and involved, using induction on r. A simpler proof for the special case that G is r-chromatic has been sketched by Kleitman [2] in his review of [1]. The object of this note is to

Mathematics Subject Classification (2000): 05C35; 05C15, 05C75 prove a structural result for maximal K_r-free graphs (i.e. graphs in which no pair of non-adjacent vertices can be joined by an edge without creating a K_r), which gives very simple proof of Theorem 1.

The crucial substructure is that of a *5-wheel-like graph* $W_{r,k}$. This is a graph consisting of two cliques Q_1, Q_2 of order $r-2$, which intersect in exactly k vertices, where $0 \le k < r-2$, together with a vertex v, adjacent to all vertices of Q_1 and Q_2 and an edge w_1w_2, where w_1 is adjacent to the vertices in Q_1 and w_2 is adjacent to the vertices in Q_2 (see Figure 1). So $W_{3,0}$ is the 5-cycle, and $W_{4,1}$ is the wheel with 5 spokes. We call the vertex v the top and the edge w_1w_2 the bottom of $W_{r,k}$.

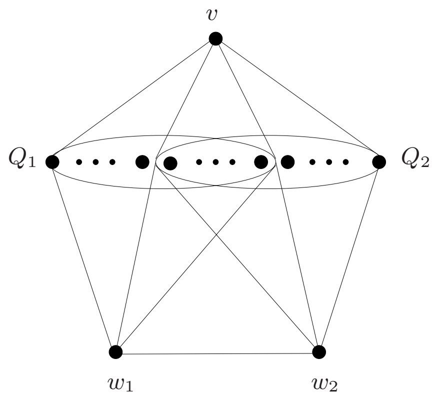

Figure 1. A 5-wheel-like graph.

While we consider r as fixed, we try to minimize the order (i.e. maximize k) of the 5-wheel-like subgraphs of a graph G (if any). It will turn out that every maximal K_r-free graph with chromatic number at least r has a (not necessarily induced) 5-wheel-like subgraph $W_{r,k}$. If for every such subgraph k is small then the minimum degree has to be relatively small as well. The result is the generalization to graphs of larger clique number of the observation that every maximal triangle-free graph is either complete bipartite or contains a 5-cycle.

Proposition 1. *Let G be a maximal K_r-free graph. Then G is either complete $(r-1)$-partite, or G contains a 5-wheel-like subgraph.*

Proof. If G is complete multipartite then G is $(r-1)$-partite and hence G cannot contain a graph $W_{r,k}$, since $W_{r,k}$ is r-chromatic. Otherwise the complement of G is not a collection of vertex disjoint cliques and therefore contains an induced path P_3 on 3 vertices. This implies that G contains an induced $\overline{P}_3$, i.e. a vertex v and an edge w_1w_2 such that $w_1, w_2 \notin N(v)$. Since G is maximal K_r-free, there must be a clique Q_i of order $r-2$ in the subgraph induced by $N(v) \cap N(w_i)$ for $i=1,2$. Otherwise the edge vw_i could be added to G without creating K_r. Since G is K_r-free we get $Q_1 \neq Q_2$, so G contains a 5-wheel-like subgraph. ∎

Proposition 2. *Let G be a K_r-free graph containing a 5-wheel-like subgraph $W_{r,k}$ with top v and bottom w_1w_2. If $\delta(G) > (2r+k-4)n/(2r+k-1)$ then $k < r-3$ and G contains $W_{r,k+1}$, having top v and bottom w_1w_2.*

Proof. Let $W_{r,k}$ be a 5-wheel-like subgraph of minimal order $p = 2r-k-1$ of G. Let X be the vertices of G which are adjacent to all vertices in the intersection of Q_1 and Q_2. Since G is K_r-free, no vertex can be adjacent to all vertices of $W_{r,k}$, so every vertex of G has at most $p-1 = 2r-k-2$ neighbours in $W_{r,k}$. Every vertex $x \in X$ is adjacent to at most $p-3 = 2r-k-4$ vertices of $W_{r,k}$. Indeed, since G is K_r-free, x is adjacent to at most $r-2$ vertices in $V(Q_i) \cup \{w_i\}$ for $i=1,2$. If x is adjacent to exactly $r-2$ vertices in $V(Q_i) \cup \{w_i\}$ for $i=1,2$ and to v, then since x is adjacent to every vertex in $V(Q_1) \cap V(Q_2)$, it must be adjacent to every vertex of $W_{r,k}$ except for exactly one vertex $q_1 \in V(Q_1) \setminus V(Q_2)$ and $q_2 \in V(Q_2) \setminus V(Q_1)$. But then the subgraph induced by $\{x\} \cup (V(W_{r,k}) \setminus \{q_1, q_2\})$ contains K_r if $k = r-3$, or a 5-wheel-like subgraph $W_{r,k+1}$ of smaller order if $k < r-3$, which leads to a contradiction in both cases. Hence

$$\delta(G) \leq \min\{d(z) \mid z \in V(W_{r,k})\} \leq \frac{|X|(2r-k-4) + (n-|X|)(2r-k-2)}{2r-k-1}.$$

Since $|X| \geq k\delta(G) - (k-1)n$ (note that this also holds if $k=0$), we get that $\delta(G) \leq (2r+k-4)n/(2r+k-1)$. ∎

In particular, we get that every 4-chromatic maximal K_4-free graph with $\delta > 4n/7$ contains the 5-wheel $W_{4,1}$ as a subgraph.

The degree bound in Proposition 2 is best possible in the sense that there are infinitely many graphs that attain the degree bound with equality for every pair (r,k) with $0 \leq k \leq r-3$. Denote by $G[H]$ the lexicographic product of G with H (for the definition see, e.g., [4, p. 370]). Set $G = \overline{C}_{2r-2k-1} \vee K_k[\overline{K}_3]$ the join of the complement of the $(2r-2k-1)$-cycle and the lexicographic product of the complete graph K_k with the edgeless graph on three vertices. Then, for every integer $i \geq 1$, $G[\overline{K}_i]$ is maximal K_r-free, d-regular with $d = (2r+k-4)n/(2r+k-1)$, and has $W_{r,k}$ as its 5-wheel-like subgraph of smallest order.

From Proposition 2 we immediately get the proof of Theorem 1.

Proof of Theorem 1. Let G be a K_r-free graph. Choose a maximal K_r-free supergraph G' on $V(G)$. If G' is complete $(r-1)$-partite then G is $(r-1)$-colourable as well. Otherwise, by Proposition 1, G' contains a 5-wheel-like subgraph $W_{r,k}$ and Proposition 2 implies that $\delta(G') \leq (3r-7)n/(3r-4) = (1-\frac{1}{r-\frac{4}{3}})n$ since $k<r-2$, a contradiction. ∎

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