Improved Bounds for the Excluded Grid Theorem

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Theorem 1. *There is some function $f : \mathbb{N} \to \mathbb{N}$, such that for every integer $g \geq 1$, every graph of treewidth at least $f(g)$ contains the $(g \times g)$-grid as a minor.*

Some definitions We say that a collection of paths $\mathcal{P}$ is *linking* vertices of T' to vertices of T'' iff it connects different pairs of vertices and $|\mathcal{P}| = |T'| = |T''|$.

Definition 2 (Well-linkedness)**.** Given a graph G, a subset $T \subseteq V(G)$ of vertices, and a parameter $0 < \alpha \leq 1$, we say that T is *α-well-linked* in G, iff for every pair of disjoint equal-sized subsets $T', T'' \subseteq T$, there is a collection of path linking vertices of T' to vertices of T'' a with congestion at most α in G.

Definition 3 (Node-well-linked)**.** We say that a set T of vertices is *node-well-linked* in G, iff for every pair (T', T'') of disjoint equal-sized subsets of T, there is a collection of node-disjoint paths linking vertices of T' to vertices of T'' in G.

Definition 4 (Node-linkedness)**.** We say that two disjoint subsets T_1, T_2 of vertices of G are *α-linked* for $0 < \alpha \leq 1$, if for every pair $T_1' \subseteq T_1$ and $T_2' \subseteq T_2$ of equal-sized vertex subsets, there is a collection of paths linking vertices of T_1 to vertices of T_2' in G. We say that they are *node-linked*, iff for every pair $T_1' \subseteq T_1$ and $T_2' \subseteq T_2$ of equal-sized subsets, there is a collection of $|T_1'|$ node-disjoint paths connecting the vertices of T_1' to the vertices of T_2'.

Definition 5 (Path-of-Sets System (P-o-S))**.** A *Path-of-Sets System* $(\mathcal{S}, \bigcup_{i=1}^{\ell} \mathcal{P}_i)$ of width w and length ℓ in a graph G consists of:

- A sequence $\mathcal{S} = (S_1, S_2, \ldots, S_\ell)$ of disjoint vertex subsets of G, where for each $i, G[S_i]$ is connected;
- For each $1 \leq i \leq \ell$, two disjoint sets $A_i; B_i \subseteq S_i$ of w vertices each; the vertices of $A_1 \cup B_\ell$ must have degree at most 2 in G; and
- For each $1 \leq i \leq \ell$, a set $\mathcal{P}_i$ is a collection of w paths linking vertices of B_i to vertices A_{i+1}, such that all paths in $\bigcup \mathcal{P}_i$ are mutually node-disjoint, and do not contain the vertices of $\bigcup S_i$ as inner vertices.

α-weak if for all $1 \leq i \leq \ell, A_i \cup B_i$ is α-well-linked in $G[S_i]$,

good if for all $1 \leq i \leq \ell, B_i$ is 1-well-linked in $G[S_i]$, and (A_i, B_i) are $\frac{1}{2}$-linked in $G[S_i]$,

perfect if for each $1 \leq i \leq \ell, A_i$ and B_i is node-well-linked in $G[S_i]$, and (A_i, B_i) are node-linked in $G[S_i]$.

Main ingredinets

Lemma 6. *Let G be any graph with maximum vertex degree 3, and $T_1, T_2 \subseteq V(G)$ any two disjoint subset of vertices of G of cardinality k each, such that (T_1, T_2) are node-linked in G, each of T_1, T_2 is node-well-linked in G, and the degree of every vertex in $T_1 \cup T_2$ is at most 2 in G. Let $w, \ell > 1$ be integers, where ℓ is an integral power of 2, and assume that for some large enough constant $c_p, k \geq c_p w \ell^{17}$.*

Then there is a perfect Path-of-Sets System $(\mathcal{S}, \mathcal{P}_i)$ of length ℓ and width w in G. Moreover, if A_1, B_ℓ are the anchors of Path-of-Sets System, then $T_1 \subseteq T_1$ and $B_\ell \subseteq T_2$.

Theorem 7. *Let G be a graph of treewidth k. Then there is a subgraph G' of G, whose maximum vertex degree is 3, and $tw(G') = O(k/\operatorname{poly}\log k)$. Moreover, there is a set $T \subseteq V(G')$ of $\Omega(k/\operatorname{poly}\log k)$ vertices, such that T is node-well-linked in G', and each vertex of T has degree 1 in G'.*

Theorem 8. *Let G be any graph, $g > 1$ an intger, and let $(\mathcal{S}, \mathcal{P}_i)$ be a perfect Path-of-Sets System of width $w = 16g^2 + 10g$ and length $\ell = 2g(g-1)$. Then G contains the $(g \times g)$-grid as a minor.*

Extending production chain

Theorem 9. *Suppose we are given a graph G with maximum vertex degree 3, and a good Path-of-Sets System $(\mathcal{S} = (S_1, \ldots, S_\ell), \bigcup_{i=1}^{\ell-1} \mathcal{P}_i)$ of length ℓ and width w, where $w > 12000$ is an integral power of 2. Let $A_1 \subseteq S_1, B_\ell \subseteq B_\ell$ denote the anchors of the Path-of-Sets System.*

Then there is a good Path-of-Sets System $(\mathcal{S}' = (S_1, \ldots, S_{2\ell}), \bigcup_{i=1}^{2\ell-1} \mathcal{P}_i)$ of length 2ℓ and width $w/2^{17}$ in G. Moreover, if $A'_1 \subseteq S'_1, B'_{2\ell} \subseteq S'_{2\ell}$ denote the anchors of this new Path-of-Sets System, then $A'_1 \subseteq A_1$ and $B'_{2\ell} \subseteq B_\ell$.

Theorem 10. *Suppose we are given a graph G, with maximum vertex degree at most 3, and two disjoint subsets of vertices, T_1 of size k (where $k \geq 12000$ is an integral power of 2) and T_2 of size $k' = k/64$, such that the degree of every vertex in $T_1 \cup T_2$ is 1 in G, the vertices of T_1 are 1-well-linked, and (T_1, T_2) are 1/2-linked in G.*

Then there is a 2-cluster chain in G.