Approximating Nash Equilibria and Dense Subgraphs via an Approximate Version of Carathéodory's Theorem

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1 Introduction & notation

$\|x\|_p$ p-norm of a vector $x \in \mathbb{R}^d$
The number of non-zero components of a vector x - ℓ_0 "norm": $\|x\|_0 := |\{i \mid x_i \neq 0\}|$
Δ^n be the set of probability distributions over the set $[n]$
vector $v \in \mathbb{R}^n$ write $\text{Supp}(v) : \{i \mid v_i \neq 0\}$
$X = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d$: $\text{conv}(X)$ = convex hull of X
A vector $y \in \text{conv}(X)$ is said to be k *uniform* with respect to X if there exists a size k multiset S of $[n]$ such that $y = \frac{1}{k}\sum_{i \in S} x_i$. $\hat{y} \in \text{conv }(X) \Leftrightarrow \hat{(}y) = Cp, p \in \Delta^n$, columns of $C = x$.

2 Approximate Version of Carathéodory's Theorem

Theorem 1 (Khintchine Inequality)**.** *Let $r_1, r_2, \ldots, r_m$ be a sequence of i.i.d. Rademacher ± 1 random variables, i.e., $\Pr(r_i = \pm 1) = \frac{1}{2}$ for all $i \in [m]$. In addition, let $u_1, u_2, \ldots, u_m \in \mathbb{R}^d$ be a deterministic sequence of vectors. Then, for $2 \leq p < \infty$*

$$\mathbb{E}\left\|\sum_{i=1}^{m} r_i u_i\right\|_p \leq \sqrt{p}\ \left(\sum_{i=1}^{m}\|u_i\|_p^2\right)^{\frac{1}{2}}. \tag{1}$$

Theorem 2. *Given a set of vectors $X = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d$ and $\varepsilon > 0$. For every $\mu \in conv(X)$ and $2 \leq p < \infty$ there exists an $\frac{4p\gamma^2}{\varepsilon^2}$ uniform vector $\mu' \in conv(X)$ such that $\|\mu - \mu'\|_p \leq \varepsilon$. Here, $\gamma := \max_{x \in X}\|x\|_p$.*

Theorem 3. *Given a set of vectors $X = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}^d$, with $\max_{x \in X}\|x\|_\infty \leq 1$, and $\varepsilon > 0$. For every $\mu \in conv(X)$ there exists an $O\left(\frac{\log n}{\varepsilon^2}\right)$ uniform vector $\mu' \in conv(X)$ such that $\|\mu - \mu'\|_\infty \leq \varepsilon$.*

3 Computing Approximate Nash Equilibrium

Bimatrix Games. two player games specified by a pair of $n \times n$ matrices (A, B) (payoff matrices). The row player has payoff matrix A, the column player, has payoff matrix B. The strategy set for each player is $[n] = \{1, 2, \ldots, n\}$, and, if the row player plays strategy i and column player plays strategy j, then the payoffs of the two players are A_{ij} and B_{ij} respectively. We assume

$x = (x_1, \ldots, x_n)$ mixed strategy for row player (x_i = probability if i-th strategy), y mixed strategy for columns, $A_{ij}, B_{ij} \in [-1, 1]$ for all $i, j \in [n]$.

The expected payoff of the row player is $x^T Ay$ and the expected payoff of the column player is $x^T By$.

Definition 1. *A mixed strategy pair (x, y), $x, y \in \Delta^n$, is said to be*
a) Nash equilibrium if and only if:

$$x^T Ay \geq e_i^T Ay \qquad \forall\, i \in [n] \quad \textit{and} \tag{2}$$
$$x^T By \geq x^T Be_j \qquad \forall\, j \in [n]. \tag{3}$$

b) ε-Nash equilibrium if and only if:

$$x^T Ay \geq e_i^T Ay - \varepsilon \qquad \forall\, i \in [n] \quad \textit{and} \tag{4}$$
$$x^T By \geq x^T Be_j - \varepsilon \qquad \forall\, j \in [n]. \tag{5}$$

Definition 2 (s-Sparse Games)**.** *The sparsity of a game (A, B) is defined to be $s := \max\{\max_i \|C^i\|_0, 4\}$, where matrix $C = A + B$.*

Theorem 4. *Let $A, B \in [-1, 1]^{n\times n}$ be the payoff matrices of an s-sparse bimatrix game. Then, an ε-Nash equilibrium of (A, B) can be computed in time $n^{O\left(\frac{\log s}{\varepsilon^2}\right)}$.*

Bilinear program.

$$\begin{aligned} \max_{x,y,\pi_1,\pi_2} \quad & x^T Cy - \pi_1 - \pi_2 \\ \text{subject to} \quad & x^T B \leq \mathbb{1}^T \pi_2 \\ & Ay \leq \mathbb{1}\pi_1 \\ & x, y \in \Delta^n \\ & \pi_1, \pi_2 \in [-1, 1]. \end{aligned} \tag{BP}$$

Theorem 5 (Equivalence Theorem)**.** *Mixed strategy pair $(\hat{x}, \hat{y})$ is a Nash equilibrium of the game (A, B) if and only if $\hat{x}$, $\hat{y}$, $\hat{\pi}_1$, and $\hat{\pi}_2$ form an optimal solution of the bilinear program (BP), for some scalars $\hat{\pi}_1$ and $\hat{\pi}_2$. In addition, the optimal value achieved by (BP) is equal to zero and the payoffs of the row and column player at this equilibrium are $\hat{\pi}_1$ and $\hat{\pi}_2$ respectively.*

Lemma 1. *Let $x, y \in \Delta^n$ along with scalars π_1 and π_2 form a feasible solution of (BP) that achieves an objective function value more than $-\varepsilon$, i.e., $x^T Cy \geq \pi_1 + \pi_2 - \varepsilon$. Then, (x, y) is an ε-Nash equilibrium of the game (A, B).*

Remark 1. *Consider the class of games in which the p norm of the columns of matrix C is a fixed constant. A simple modification of the arguments mentioned above shows that for such games an ε-Nash equilibrium can be computed in time $n^{O\left(\frac{p}{\varepsilon^2}\right)}$.*

Algorithm 1 Algorithm for computing ε-Nash equilibrium in s-sparse games

Given payoff matrices $A, B \in [-1,1]^{n\times n}$ and $\varepsilon > 0$; Return: ε-Nash equilibrium of (A, B)

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Write s to denote the sparsity of the game (A, B) and let p = log s ( p ≥ 2 ).
Let U be the collection of all multisets of {1, 2, ..., n} of cardinality at most κ p / ε^2, where κ is a
   fixed constant.
Write C^i to denote the ith column of matrix C = A + B, for i ∈ [n].
for all multisets S ∈ U do
   Set u = (1/|S|) Σ_{i∈S} C^i.
   {u is an |S|-uniform vector in the convex hull of the columns of C.}
   Solve convex program CP(u).
   if the objective function value of CP(u) is less than ε/2 then
      Return (x, y), where x and y form an optimal solution of CP(u).
   end if
end for
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Remark 2. *Algorithm 1 can be adopted to find an approximate Nash equilibrium with large social welfare (the total payoffs of the players). Specifically, in order to determine whether there exists an approximate Nash equilibrium with social welfare more than $\alpha - \varepsilon$, we include the constraint $\pi_1 + \pi_2 \geq \alpha$ in CP(u). The time complexity of the algorithm stays the same, and then via a binary search over α we can find an approximate Nash equilibrium with near-optimal social welfare.*

4 Applications

4.1 Small Probability Games

Theorem 6. *Let $A, B \in [-1,1]^{n\times n}$ be the payoff matrices of an s-sparse bimatrix game. If (A, B) contains an m-small probability Nash equilibrium, then an ε-Nash equilibrium of the game can be computed in time $n^{O\left(\frac{t}{\varepsilon^2}\right)}$, where $t = \max\left\{2\log\left(\frac{s}{m}\right), 2\right\}$.*

4.2 Densest Subgraph

Densest Subgraph. The normalized densest k-subgraph problem (NDkS): find size-k subgraph of maximum density (number of edges in the subgraph divided by k^2).

Theorem 7. *Let G be a graph with n vertices and maximum degree d. Then, an ε-additive approximation of* NDkS over G can be determined in time $n^{O\left(\frac{\log d}{\varepsilon^2}\right)}$.

Theorem 8. *Let G be a graph with n vertices and maximum degree d. Then, there exists an algorithm that runs in time $n^{O\left(\frac{\log d}{\varepsilon^2}\right)}$ and computes a $k \times k$-bipartite subgraph of density at least $\rho(S^*, T^*) - \varepsilon$.*

4.3 Approximating Colorful Carathéodory and Tverberg's Theorem

Theorem 9 (Colorful Carathéodory Theorem)**.** *Let* $X_1, X_2, \ldots, X_{d+1}$ *be* $d+1$ *sets in* $\mathbb{R}^d$ *and vector* $\mu \in \cap_i conv(X_i)$. *Then, there exists* $d+1$ *vectors* $x_1, x_2, \ldots, x_{d+1}$ *such that* $x_i \in X_i$ *for each* i *and* $\mu \in conv(\{x_1, x_2, \ldots, x_{d+1}\})$.

Theorem 10 (Tverberg's Theorem)**.** *Any set of* $(r-1)(d+1)+1$ *vectors* $X \subset \mathbb{R}^d$ *can be partitioned into* r *pairwise disjoint subsets* $X_1, X_2, \ldots, X_r \subseteq X$ *such that their convex hulls intersect:* $\cap_{i=1}^{r} conv(X_i) \neq \phi$.

Definition 3 (Concurrently ε close)**.** *Sets* $V_1, V_2, ..., V_r \subset \mathbb{R}^d$ *are said to be concurrently* ε *close under the* p*-norm distance if there exists a vector* $\mu \in \mathbb{R}^d$ *such that* $\inf_{v \in conv(V_i)} \|\mu - v\|_p \leq \varepsilon$, *for all* $i \in [r]$.

Theorem 11. *Let norm* $p \in [2, \infty)$ *and parameter* $\varepsilon > 0$. *Then, any set of* $(r-1)(d+1)+1$ *vectors* $X \subset \mathbb{R}^d$ *can be partitioned into* r *pairwise disjoint subsets* $X'_1, X'_2, \ldots, X'_r \subseteq X$ *that are concurrently* ε *close under the* p*-norm distance and satisfy* $|X'_i| = O\left(\frac{p\gamma^2}{\varepsilon^2}\right)$, *for all* $i \in [r]$. *Here,* $\gamma := \max_{x \in X} \|x\|_p$.