1 Notations and definitions

Definition 1.1. 1. A one round two prover game $G$ is specified by a bipartite graph with vertex sets $U$ and $V$ such that each edge $(u,v)$ is endowed with a constraint $\pi_{(u,v)} \subseteq \Sigma \times \Sigma$.

$$\text{val}(G) := \max_{f,g} \text{val}(G, f, g) := \max_{f,g} \Pr_{(u,v)}[(f(u), g(v)) \in \pi_{u,v}].$$

2. A game is called projection game if for every edge $(u,v)$ and for every possible answer of Bob $\beta \in \Sigma$, there exists a single answer $\alpha \in \Sigma$ for $A$ such that the constraint $\pi_{(u,v)}$ is satisfied.

3. In the $k$-folded parallel repetition of the game $G^{\otimes k}$ the referee chooses $k$ edges $(u_1, v_1), \ldots, (u_k, v_k)$ independently from $G$, and sends $(u_1, \ldots, u_k)$ to $A$, and $(v_1, \ldots, v_k)$ to $B$. Each of the players answers with a $k$-tuple string and they win if their answers satisfies each of the constraints on these edges.

Theorem 1.2 (Parallel Repetition Theorem). For any projection game $G$, if $\text{val}(G) = 1 - \epsilon$ then $\text{val}(G^{\otimes k}) \leq (1 - c\epsilon^2)^{k/2}$.

2 Linear Algebra Notations:

Denote by $L(U \times \Sigma)$ the set of real-valued functions defined on $U \times \Sigma$. An assignment for $A$ is specified by a function $g \in L(U, \Sigma)$ satisfying (i) $g(u, \alpha) \geq 0$, (ii) for every $u \in U$: $\sum_{\alpha} g(u, \alpha) = 1$ (Note that $A$ maybe non-deterministic, on a query $u$ she outputs $\alpha$ with probability $\frac{g(u, \alpha)}{\sum_{\alpha'} g(u, \alpha')}$.)

For every projection game $G$ we define a linear operator from $L(V \times \Sigma)$ to $L(U \times \Sigma)$ by:

$$Gf(u, \alpha) = \mathbb{E}_{v \mid u} \sum_{\beta : \alpha \leftarrow \beta} f(v, \beta), \text{ where } \alpha \leftarrow \beta \text{ means that } (\alpha, \beta) \in \pi(u, v).$$

We define an inner product $\langle g, g' \rangle = \mathbb{E}_{u} \sum_{\alpha} g(u, \alpha) g'(u, \alpha)$.

Claim 2.1. 1. $Gf(u, \alpha) \geq 0$, $\sum_{\alpha} f(u, \alpha) \leq 1$.

2. $\text{val}(G, f, g) = \langle g, Gf \rangle$.

3. For any two projections games $G, H$, let $f \in L(U \times \Sigma \times U' \times \Sigma')$ we define:

$$G \otimes H f(u, \alpha, u', \alpha') = \mathbb{E}_{v \mid u, v' \mid u'} \sum_{\beta, \alpha \leftarrow \beta, \alpha' \leftarrow \beta'} f(v, \beta, v', \beta'),$$

then $\text{val}(G \otimes H, f, g) = \langle g, (G \otimes H)f \rangle$. 


Definition 2.2 (Collision Value and Symmetrized Game). Let $G$ be a projection game, and let $f$ be an assignment to $B$:

$$\|Gf\| = (Gf, Gf)^{1/2}.$$  

$$\|G\| = \max_f \|Gf\|$$

Given a projection game $G$, define $G_{\text{sym}}$ to be the following game. A referee chooses $u \in U$, and then $v, v'\mid u$ independently, and sends $v$ to $A$ and $v'$ to $B$. $A$ responds with $\beta$ and $B$ with $\beta'$. The referee accepts iff there exists $\alpha$ such that $\alpha \leftarrow \beta$ and $\alpha \leftarrow \beta'$ (the values collide).

Consider the (weighted) graph induced by the $G_{\text{sym}}$, i.e. $V$ is the set of vertices. The probability we assign to $(v, v')$ is the probability induced by first picking $u \in U$ and then $v, v' \sim u$. $G$ is called $\lambda$-expander if the induced graph of $G_{\text{sym}}$ is $\lambda$-expander.

Claim 2.3. 1. $\text{val}(G_{\text{sym}}) = \|G\|$.  
2. $\text{val}(G) \leq \|G\| \leq \text{val}(G)^{1/2}$.  
3. $\text{val}(\|G \otimes H\|) \leq \text{val}(\|H\|)$.

3 Proof of the Main Result

Lemma 3.1 (Main Lemma). Let $G, H$ be any projection games satisfying $\text{val}(G) = 1 - \epsilon$, then:

$$\|G \otimes H\|^2 \leq (1 - \epsilon^2) \|H\|^2.$$  

For every projection game we define the following quantities:

$$\rho_G := \sup_H \frac{\|G \otimes H\|}{\|H\|},$$

$$\lambda_+(G) := \max_{h \geq 0} \frac{\|G h\|}{\|T h\|}.$$  

Where $T$ is the following trivial game on which the referee picks $(v, v') \in V$ and accepts iff $A$ answers 1.

Lemma 3.2 (Main Technical Lemma). 1. Let $G, H$ be any projection games, then: $\|G \otimes H\| \leq \lambda_+(G) \|H\|$.  
2. Let $G$ be a $\lambda$-expander projection game, then if $\lambda_+(G) > 1 - \delta$ then $\|G\| > 1 - O(\delta^2/\lambda)$.  

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