ENTROPY X ... finite set ; μ ... measure on X

$$H(\mu) := -\sum_{x\in X} \log(\mu(x))\cdot\mu(x)$$

RELATIVE ENTROPY (X,μ) ; (X,ν) ... μ,ν two measures on X

$$D(\mu\|\nu) := \sum_{x\in X}\left[\log\mu(x) - \log\nu(x)\right]\mu(x)$$

where
- summand is 0 whenever $\mu(x)=0$
- $\nu(x)=0$ implies $\mu(x)=0$

Measure preserving map $\Psi : X \to Y$ is m.p. if $\mu(\Psi^{-1}(y)) = \nu(y)$

Joint factor $(X_i,\mu_i)_{i=1}^{3}$ ... finite probability spaces

Assume that $\left.\begin{array}{l}\Psi_1 : X_1 \to X_3\\ \Psi_2 : X_2 \to X_3\end{array}\right\}$ are measure preserving maps.

Then X_3 is a joint factor of (X_1,μ_1) and (X_2,μ_2)

Coupling of (X_1,μ_1), and (X_2,μ_2) over (X_3,μ_3)

$$X_4 := \{(x_1,x_2)\in X_1\times X_2 : \Psi_1(x_1)=\Psi_2(x_2),\ \mu_3(\Psi_1(x_1))\neq 0\}.$$

The measure μ is called a <u>coupling</u>.. if the projections $\Pi_i : X_4 \to X_i$ are measure preserving on (X_4,μ).

Conditionally independent Coupling Let μ_4 be the measure on X_4 defined by

$$\boxed{\mu_4((x_1,x_2)) := \frac{\mu_1(x_1)\cdot\mu_2(x_2)}{\mu_3(\Psi_1^{-1}(x_1))}}$$

Then it is clear that projections are measure preserving

Lemma: Among couplings, the conditionally independent coupling minimizes the relative entropy with respect to a conditionally independent coupling.

PIE-type expression

$$D(\mu_4\|\nu_4) = D(\mu_1\|\nu_1) + D(\mu_2\|\nu_2) - D(\mu_3\|\nu_3)$$

Vertex factor μ … prob. distribution on $\mathrm{Hom}(H,G)$
$\beta : S \to V(H)$ be an injective map
Map $\phi \mapsto \phi \circ \beta$ [where $\phi \in \mathrm{Hom}(H,G)$]
defines a factor of $(\mathrm{Hom}(H,G), \mu)$.
Denote this factor by $(V(G)^S, \mu|_\beta)$ and call it <u>vertex factor</u>

$S \subseteq V(H)$ by $\mu|_S$ denote $\mu|_\beta$ with $\beta: S \to S$ identity

Coupling $(\mathrm{Hom}(H_1,G),\mu_1), (\mathrm{Hom}(H_2,G),\mu_2)$ … prob. spaces
$\{\beta_i : [n] \to V(H_i)\}_{i=1}^{2}$ such that $\mu_3 = \mu_1|_{\beta_1} = \mu_2|_{\beta_2}$

denote by $C(\mu_1,\mu_2,\beta_1,\beta_2)$ the conditionally independent coupling of μ_1 and μ_2 over μ_3.

Probability Scheme: of a graph H is a function f on the set of finite graphs whose value $f(G)$ is ~~the~~ a probability distribution on $\mathrm{Hom}(H,G)$
We say that H is the <u>frame</u> of the scheme f.

Joint factor f_i … prob. scheme for H_i
$\beta_i : [n] \to V(H_i)$ … two labelings such that $\overbrace{f_1(G)|_{\beta_1} = f_2(G)|_{\beta_2}}^{\forall G}$
β_1 and β_2 define a <u>joint vertex factor</u> of f_1 and f_2.

Coupling The conditionally independent coupling $g = C(f_1, f_2, \beta_1, \beta_2)$ of f_1 and f_2 is the function g whose value on G is $C(f_1(G), f_2(G), \beta_1, \beta_2)$

The frame of g: is the graph obtained by the disjoint union of H_1, H_2 and identifying the vertices with the same label.

$\mathcal{A}_1$ Let $\mathcal{A}_1$ denote the smallest set of prob. schemes containing the scheme $G \to \mathcal{J}(e,G)$ and closed with respect to conditionally independent couplings <u>over independent vertex sets</u>.

Lemma: Every element in $\mathcal{A}_1$ is a family of witness measures.