Tverberg plus constraints

P. V. M. Blagojević, F. Frick & G. M. Ziegler

April 2, 2015

presented by Vojtěch Kaluža

Definition (Simplex): *A d-dim* ***simplex*** *is a convex hull of $d+1$ affinely independent vectors. For example:* $\operatorname{conv}(\{0, e_1, \ldots, e_d\})$ *in* $\mathbb{R}^d$.

- *Every subset of its vertices determines its proper face.*

Definition (Simplicial complex): *Let K be a set of simplices such that*

(i) $\forall \sigma \in K$ *and for every face* τ *of* σ *also* $\tau \in K$

(ii) $\forall \sigma, \tau \in K$ *the intersection* $\sigma \cap \tau$ *is a face of both* σ *and* τ

Then K is a ***simplicial complex****.*

Definition (Subcomplex): *Let K be a simplicial complex. The set $L \subseteq K$ is its* ***subcomplex*** *if it is also a simplicial complex. ($\Leftrightarrow$ The property (i) from the above definition holds for L.)*

Theorem (Affine Tverberg Theorem): *Let $d \geq 1$ and $r \geq 2$ be integers, and $N = (r-1)(d+1)$. For any affine map $f\colon \Delta_N \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Definition (Tverberg (r)-partition): *The set of r pairwise disjoint simplices $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$ is called a* ***Tverberg r-partition****.*

Theorem (Topological Tverberg Theorem): *Let $r \geq 2$, $d \geq 1$, and $N = (r-1)(d+1)$. If r is a prime power, then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Lemma (Key Lemma 1): *Let $r \geq 2$ be a prime power, $d \geq 1$, and $c \geq 0$. Let $N \geq N_c := (r-1)(d+1+c)$ and let $f : \Delta_N \to \mathbb{R}^d$ and $g : \Delta_N \to \mathbb{R}^c$ be continuous. Then there are r points $x_i \in \sigma_i$, where $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint faces of Δ_N with $g(x_1) = \cdots = g(x_r)$ and $f(x_1) = \cdots = f(x_r)$.*

Definition (Tverberg unavoidable subcomplex): *Let $r \geq 2$, $d \geq 1, N \geq r-1$ be integers and $f : \Delta_N \to \mathbb{R}^d$ a continuous map with at least one Tverberg r-partition. Then a subcomplex $\Sigma \subseteq \Delta_N$ is* ***Tverberg unavoidable*** $\Longleftrightarrow$ *For every Tverberg partition $\sigma_1, \ldots, \sigma_r$ for f there is at least one face σ_j that lies in Σ.*

Definition (k-skeleton): *The* **k-*skeleton*** $K^{(k)}$ *of a simplicial complex K is its subcomplex consisting of all faces of dimension at most k.*

Lemma (Key examples): *Let $d \geq 1$, $r \geq 2$, and $N \geq r-1$. Assume that the continuous map $f : \Delta_N \to \mathbb{R}^d$ has a Tverberg r-partition. Then the following holds:*

1. *The induced subcomplex (simplex) $\Delta_{N-(r-1)}$ on $N-r+2$ vertices of Δ_N is Tverberg unavoidable.*

2. *For any set S of at most $2r-1$ vertices in Δ_N the subcomplex of faces with at most one vertex in S is Tverberg unavoidable.*

3. *If k is an integer such that $r(k+2) > N+1$, then the k-skeleton $\Delta_N^{(k)}$ of Δ_N is Tverberg unavoidable.*

4. *If $k \geq 0$ and s are integers such that $r(k+1)+s > N+1$ with $0 \leq s \leq r$, then the subcomplex $\Delta_N^{(k-1)} \cup \Delta_{N-(r-s)}^{(k)}$ of Δ_N is Tverberg unavoidable.*

Lemma (Key Lemma 2): *Let $r \geq 2$ be a prime power, $d \geq 1$.*

a) Let $N \geq N_1 = (r-1)(d+2)$. Assume that $f : \Delta_N \to \mathbb{R}^d$ is continuous and that the subcomplex $\Sigma \subseteq \Delta_N$ is Tverberg unavoidable for f. Then there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N, all of them contained in Σ, such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

b) Let $c \geq 1$, and $N \geq N_c = (r-1)(d+1+c)$. Let $f : \Delta_N \to \mathbb{R}^d$ be continuous and let $\Sigma_1, \Sigma_2, \ldots, \Sigma_c \subseteq \Delta_N$ be Tverberg unavoidable subcomplexes for f. Then there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ in $\Sigma_1 \cap \cdots \cap \Sigma_c$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

Definition (Rainbow complex, rainbow simplex): *Suppose that the vertices of Δ_N are colored. Denote by $R \subseteq \Delta_N$ the* ***rainbow complex****, i.e., the subcomplex of faces that have at most one vertex of each color class. These faces are called* ***rainbow faces****.*

Theorem (Variant of colored Tverberg)(5.4): *Let $r \geq 2$ be a prime power, $d \geq 1$, $c \geq \lceil \frac{r-1}{r} d \rceil + 1$, and $N \geq N_c = (r-1)(d+1+c)$. Let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of Δ_N are divided into c color classes, each of cardinality at most $2r-1$, then there are r pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Theorem (Optimal colored Tverberg): *Let $r \geq 2$ be a prime, $d \geq 1$, and $N \geq N_0 = (r-1)(d+1)$. Let the vertices of Δ_N be colored by $m+1$ colors $C_0, \ldots, C_m$ with $|C_i| \leq r-1$ for all i. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Theorem (Generalized optimal colored Tverberg)(9.2): *Let $r \geq 2$ be a prime, $d \geq 1$, $\ell \geq 0$, and $k \geq 0$. Let the vertices of Δ_N be colored by $\ell + k$ colors $C_0, \ldots, C_{\ell+k-1}$ with $|C_0| \leq r-1, \ldots, |C_{\ell-1}| \leq r-1$ and $|C_\ell| \geq 2r-1, \ldots, |C_{\ell+k-1}| \geq 2r-1$, where $|C_0| + \cdots + |C_{\ell-1}| > (r-1)(d-k+1) - k$. Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Theorem (van Kampen-Flores): *Let $d \geq 2$ be even. Then for every continuous map $f : \Delta_{d+2} \to \mathbb{R}^d$ there are two disjoint faces $\sigma_1, \sigma_2 \subset \Delta_{d+2}$ of dimension at most $\frac{d}{2}$ in Δ_{d+2} with $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.*

Theorem (Generalized van Kampen-Flores)(6.2): *Let $r \geq 2$ be a prime power, $2 \leq j \leq r$, $d \geq 1$, and $k < d$ such that there is an integer $m \geq 0$ that satisfies*

$$(r-1)(m+1) + r(k+1) \quad \geq \quad (N+1)(j-1) \quad > \quad (r-1)(m+d+2).$$

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r j-wise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N with $\dim \sigma_i \leq k$ for $1 \leq i \leq r$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

Theorem (Generalized and sharpened van Kampen-Flores): *Let $r \geq 2$ be a prime power, $2 \leq j \leq r$, $d \geq 1$, and $k \leq N$ such that*

$$k \geq \tfrac{r-1}{r} d \quad \text{and} \quad N+1 > \tfrac{r-1}{j-1}(d+2).$$

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r j-wise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N, with $\dim \sigma_i \leq k$ for $1 \leq i \leq r$, such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.