The Parameterized Complexity of k-Biclique

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Definition 1. Let $\Sigma = \{0, 1\}$. A *parameterized problem* is a pair (Q, κ) consisting of a classical problem $Q \subseteq \Sigma^*$ and a polynomial-time computable parameterization $\kappa : \Sigma^* \to \mathbb{N}$.

An algorithm is an *fpt-algorithm with respect to a parameterization* κ if for every $x \in \Sigma^*$ the running time of the algorithm on x is bounded by $f(\kappa(x)) \cdot |x|^{O(1)}$ for a computable function $f : \mathbb{N} \to \mathbb{N}$.

A parameterized problem is *fixed-parameter tractable* (belongs to he class FPT) if it has an fpt-algorithm.

Let (Q, κ) and (Q', κ') be two parameterized problems. An *fpt-reduction* from (Q, κ) to (Q', κ') is a mapping $R : \Sigma^* \to \Sigma^*$ such that:

1. For every $x \in \Sigma^*$ we have $x \in Q$ if and only if $R(x) \in Q'$.
2. R is computable by an fpt-algorithm.
3. There is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that $\kappa'(R(x)) \leq g(\kappa(x))$ for all $x \in \Sigma^*$.

Parameterized-Biclique **Input:** A graph G, an integer k **Parameter:** k **Question:** Is there a subgraph $K_{k,k}$ in G?

Theorem 2. *Parameterized-Biclique is $W[1]$-hard.*

Reduction

Theorem 3. *For any n vertices, graph G, and positive integer k with $n^{\frac{6}{k+6}} > (k+6)!$ we can compute a graph G' in $f(k) \cdot n^{O(1)}$-time such that G' contains a $K_{k',k'}$ if and only if G contains a K_k, where $k' = \Theta(k!)$.*

Theorem 4. *For any n vertices, graph G, and positive integer k with $n \gg k$, we can compute a graph G' in $f(k) \cdot n^{O(1)}$-time such that, with high probability, G' contains a K_{k^2,k^2}, if and only if G contains a K_k.*

Definition 5 ((n, k, l, h)-threshold property)**.** Suppose that $G = (A \dot\cup B, E)$ is a bipartite graph with $A = V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_n$ and $h > l$. We say that G has the (n, k, l, h)-threshold property if it satisfies:

(T1) Every $k + 1$ distinct vertices in A have at most l common neighbors in B,

(T2) For every k distinct indices $i_1, i_2, \ldots, i_k$ there exist $v_{i_1} \in V_{i_1}, \ldots, v_{i_k} \in V_{i_k}$ such that $v_{i_1}, \ldots, v_{i_k}$ have at least h common neighbors in B.

Definition 6 (the "neighborhood" of a vector of vertices)**.** In a bipartite graph $G = (A \dot\cup B, E)$, let us have $\vec{v} = (v_1, \ldots, v_t)$, where $v_1, \ldots, v_t \in A$. We define $N(\vec{v}) = \{u \in B : v_1 u \in E, \ldots, v_t u \in E\}$.

Lemma 7 (reduction)**.** *We are given an (n, k, l, h)-threshold bipartite graph of size $f(k) \cdot n^{O(1)}$. Let $s = \binom{k}{2}$. For any n vertices and graph G on n vertices we can construct a new graph $H = (A \dot\cup B, E)$ in $f(k) \cdot n^{O(1)}$-time such that*

(H1) if $K_k \subseteq G$ then $\exists \vec{v} \in \binom{A}{s}$ such that $|N(\vec{v})| \geq h$;

(H2) if $K_k \not\subseteq G$ then $\forall \vec{v} \in \binom{A}{s}$ we have $|N(\vec{v})| \leq l$.

Lemma 8. *For $k, n \in \mathbb{N}^+$ with $k = 6l - 1$ for some $l \in \mathbb{N}^+$ and $\lceil (n+1)^{\frac{6}{k+1}} \rceil > (k+1)!$, the bipartite graph with $(n, k, (k+1)!, \lceil (n+1)^{\frac{6}{k+1}} \rceil)$-threshold property can be computed in $f(k) \cdot n^{O(1)}$-time.*

Lemma 9. *For $t = s^2$ and $n \gg t$ we can compute in $f(k) \cdot n^{O(1)}$-time a bipartite random graph satisfying the $(n, s, t-1, t)$-threshold property almost surely.*

Probabilistic Construction

Lemma 10. *For any $0 < \alpha < \beta < 1$, $\varepsilon = \frac{1}{s}$, $t = (1-\alpha)s(1+s) + 2$ and $N \gg t$, the graph $G_B\left(N, N^{-\frac{(s+1+t+\varepsilon)}{(s+1)t}}\right)$ satisfies the $(N^{1-\beta}, s, t-1, t)$-threshold property almost surely.*

Explicit Construction

Definition 11 (Paley-type Graph)**.** For any prime power $q = p^t$ and $d \mid q - 1$, $G(q, d) := (A \dot\cup B, E)$ is a Paley-type bipartite graph with

1. $A = B = GF^{\times}(q)$ (where $GF^{\times}(q)$ is the multiplicative group of $GF(q)$);
2. $\forall x \in A, y \in B :\ xy \in E$ if and only if $(x + y)^{\frac{q-1}{d}} = 1$.