Counting independent sets in graphs

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We present an elementary, yet quite powerful, method of enumerating independent sets in graphs. We illustrate this method with several applications.

The method (the Kleitman–Winston algorithm):

- For a graph G, let $\mathcal{I}(G)$ denote the family of all independent sets in G, let $i(G) := |\mathcal{I}(G)|$, and let $\alpha(G)$ be the largest cardinality of an element of $\mathcal{I}(G)$, usually called the *independence number* of G. For $m \in \mathbb{N}$, let $i(G, m)$ be the number of independent sets in G that have precisely m elements.
- For $A \subseteq V(G)$, let $e_G(A)$ denote the number $|E(G[A])|$.
- Let G be a graph with a fixed total order $\prec$ on $V(G)$. For every $A \subseteq V(G)$, the *max-degree ordering of* A is the ordering $(v_1, \ldots, v_{|A|})$ of all elements of A, where for each $j \in \{1, \ldots, |A|\}$, v_j is the maximum-degree vertex in $G[A \setminus \{v_1, \ldots, v_{j-1}\}]$. The ties are broken by giving preference to vertices that come earlier in $\prec$.
- **The algorithm:** Suppose a graph G, an $I \in \mathcal{I}(G)$, and an integer $q \leq |I|$ are given. Set $A := V(G)$ and $S := \emptyset$. For $s = 1, \ldots, q$, do the following:

  (a) Let $(v_1, \ldots, v_{|A|})$ be the max-degree ordering of A.
  (b) Let j_s be the minimal index j such that $v_j \in I$.
  (c) Move v_{j_s} from A to S.
  (d) Delete $v_1, \ldots, v_{j_s - 1}$ and $N_G(v_{j_s}) \cap A$ from A.

  Output $(j_1, \ldots, j_q)$ and $A \cap I$.
- For each output sequence $(j_1, \ldots, j_q)$ and every $s \in [q]$, denote by $A(j_1, \ldots, j_s)$ and $S(j_1, \ldots, j_s)$ the sets A and S at the end of the sth iteration of the algorithm (run on some input I that produces this particular sequence $(j_1, \ldots, j_q)$), respectively.

  Lemma 1. *Let G be a graph on n vertices and assume that an integer q and reals R and $\beta \in [0,1]$ satisfy $R \geq (1-\beta)^q n$. Suppose that the number of edges induced in G by every set $U \subseteq V(G)$ with $|U| \geq R$ satisfies $e_G(U) \geq \beta\binom{|U|}{2}$. Then, for every integer $m \geq q$, $i(G, m) \leq \binom{n}{q}\binom{R}{m-q}$.*

Applications:

- **Independent sets in regular graphs**

  Theorem 2 (A. A. Sapozhenko, 2001)**.** *There is an absolute constant C such that every n-vertex d-regular graph G satisfies*
  $$i(G) \leq 2^{\left(1 + C\sqrt{\frac{\log d}{d}}\right)\frac{n}{2}}.$$
- **Sum-free sets**

  A set A of elements of an abelian group is called *sum-free* if there are no $x, y, z \in A$ satisfying $x + y = z$.

  Theorem 3 (N. Alon, 1991)**.** *The set $[n]$ has at most $2^{(1/2+o(1))n}$ sum-free subsets.*
- **The number of C_4-free graphs**

  Call a graph C_4*-free* if it does not contain a cycle of length four and let $\mathrm{ex}(n, C_4)$ denote the maximum number of edges in a C_4-free graph with n vertices. Let $f_n(C_4)$ be the number of (labeled) C_4-free graphs on the vertex set $[n]$.

Theorem 4 (D. J. Kleitman and K. J. Winston, 1982)**.** *There is a positive constant C such that*

$$\log_2 f_n(C_4) \le Cn^{3/2}.$$

- **Independent sets in regular graphs without small eigenvalues**

Let $\lambda_1 \ge \cdots \ge \lambda_n$ be the eigenvalues of the adjacency matrix of G. We use $\lambda(G)$ to denote the smallest eigenvalue λ_n of G.

Lemma 5 (N. Alon and F. R. K. Chung, 1988)**.** *Let G be an n-vertex d-regular graph. For all $A \subseteq V(G)$,*

$$2e_G(A) \ge \frac{d}{n}|A|^2 + \frac{\lambda(G)}{n}|A|(n - |A|).$$

Theorem 6 (N. Alon, J. Balogh, R. Morris, W. Samotij, 2014)**.** *For every $\varepsilon > 0$, there exists a constant C such that the following holds. If G is an n-vertex d-regular graph with $\lambda(G) \ge -\lambda$, then*

$$i(G, m) \le \binom{\left(\frac{\lambda}{d+\lambda} + \varepsilon\right) n}{m},$$

provided that $m \ge Cn/d$.

- **Roth's theorem in random sets**

Given a positive δ, we shall say that a set $A \subseteq \mathbb{Z}$ is δ-*Roth* if each $B \subseteq A$ satisfying $|B| \ge \delta|A|$ contains 3-term arithmetic progression (3-term AP). A theorem of Roth asserts that for every $\delta > 0$ there exists an n_0 such that the set $[n]$ is δ-Roth whenever $n \ge n_0$.

Theorem 7 (Y. Kohayakawa, T. Luczak, V. Rödl, 1996)**.** *For every $\delta > 0$, there exists a constant C such that if $C\sqrt{n} \le m \le n$, then the probability that a uniformly chosen random m-element subset of $[n]$ is δ-Roth tends to 1 as $n \to \infty$.*

Theorem 8. *For every positive ε, there exists a constant D such that if $D\sqrt{n} \le m \le n$,*

$$|\{A \subseteq [n] \colon |A| = m \text{ and } A \text{ contains no 3-term AP}\}| \le \binom{\varepsilon n}{m}.$$

Proposition 9 (P. Varnavides, 1959)**.** *For every $\delta > 0$ there exist an integer n_0 and $\beta > 0$ such that if $n \ge n_0$, then every set of at least δn integers from $[n]$ contains at least βn^2 3-term APs.*