Polynomiality for Bin Packing with a Constant Number of Item Types

Michel X. Goemans, Thomas Rothvoß
Presented by Martin Böhm
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Bin packing

D(Bin Packing):

Input: A pair of vectors (s, a), where $s_1, s_2, s_3, \ldots s_d$ are *item types*, i.e. all possible sizes of our input items ($s_i \in [0, 1]$) and $a_1, a_2, \ldots, a_d$ are *item multiplicities*, i.e. how many items of each item type we need to pack ($a_i \in \mathbb{Z}_{\geq 0}$).

Goal: Find a minimum number of *bins* of capacity 1 such that all items are packed.

We are only considering a constant number of item types d.

We can look at Bin Packing also in this manner:

Input: A pair of vectors (s, a) as before. From these two vectors, define a *configuration space* $\mathbb{P} \equiv \{x \in \mathbb{Z}^d_{\geq 0} | s^T x \leq 1\}$. An element x in the configuration space represents one valid packing of a bin.

Goal: Select a minimum number of vectors in $\mathbb{P}$ such that we use all items with respect to their multiplicities, i.e. the vectors of configuration space we use sum up to a:

$$\min \left\{ \sum_i \lambda_i | \sum_{x \in \mathbb{P}} \lambda_x \cdot x = a; \lambda \in \mathbb{Z}^{\mathbb{P}}_{\geq 0} \right\}.$$

Note: Even for fixed d, both $\mathbb{P}$ and λ_x will be exponentially large.

T(Main result): For any Bin Packing instance (s, a), an optimum integral solution can be computed in time $\mathrm{O}(\log \Delta)^{2^{\mathrm{O}(d)}}$, where Δ is the largest integer appearing in the denominator s_i or in a multiplicity a_i.

The polyhedral cookbook

D: Given a set $X \subseteq \mathbb{R}^d$, we define a *convex cone* as $\mathrm{cone}(X) \equiv \{\sum_{x \in X} \lambda_x x | \lambda_x \geq 0 \forall x \in X\}$ and an *integer cone* as $\mathrm{intcone}(X) = \{\sum_{x \in X} \lambda_x x | \lambda_x \in \mathbb{Z}_{\geq 0} \forall x \in X\}$.

D: For a polytope $P = \{x \in \mathbb{R}^d | Ax \leq b\}$, we define $\mathrm{enc}(P)$ as the number of bits that it takes to write down the inequalities defining P.

D: For a vector λ we define *support* $\mathrm{supp}(\lambda)$ as the non-zero indices of λ.

D: Define a d-dimensional *parallelepiped* Π with center v_0 as

$$\Pi = \left\{ v_0 + \sum_{i=1}^{k} \mu_i v_i : |\mu_i| \leq 1 \right\}.$$

Usually we assume that parallelepipeds have linearly independent vectors v_i.

T(Finding conic combinations): Given polytopes $P, Q \subseteq \mathbb{R}^d$, one can find a $y \in \mathrm{intcone}(P \cap \mathbb{Z}^d) \cap Q$ and a vector $\lambda \in \mathbb{Z}^{P \cap \mathbb{Z}^d}_{\geq 0}$ such that $y = \sum_{x \in P \cap \mathbb{Z}_d} \lambda_x x$ in time $\mathrm{enc}(P)^{2^{\mathrm{O}(d)}} \cdot \mathrm{enc}(Q)^{\mathrm{O}(1)}$, or decide that no such y exists. Moreover, $|\mathrm{supp}(\lambda)|$ is upper bounded by 2^{2d+1}.

The previous theorem can be proven using the Structure Theorem, stated as follows:

T(Structure Theorem): Let $P = \{x \in \mathbb{R}^d | Ax \leq b\}$ be a polytope with $A \in \mathbb{Z}^{m \times d}$, $b \in \mathbb{Z}^m$ such that all coefficients are absolute-bounded by Δ. Then there exists a set $X \subseteq P \cap \mathbb{Z}^d$ of size $|X| \leq N \equiv m^d d^{\mathrm{O}(d)} (\log \Delta)^d$ that can be computed in time $N^{\mathrm{O}(1)}$ with the following property:

For any vector $a \in \mathrm{intcone}(P \cap \mathbb{Z}^d)$ there exists an integral vector $\lambda \in \mathbb{Z}^{P \cap \mathbb{Z}^d}_{\geq 0}$ such that $\sum_{x \in P \cap \mathbb{Z}^d} \lambda_x \cdot x = a$ and

1. $\lambda_x \in \{0, 1\}$ for all x outside X, that is $x \in (P \cap \mathbb{Z}^d) \setminus X$.
2. $|supp(\lambda) \cap X| \leq 2^{2d}$
3. $|supp(\lambda) \setminus X| \leq 2^{2d}$.

The recipe

The key idea behind the Structure Theorem is as follows:

- Split the polytope into polynomially many full-dimensional cells. The cells are not equicardinal, their sizes are chosen strategically.
- For each cell, we do the following:
  - We fix an arbitrary integral point of the cell.
  - We envelop all integral points of the cell by a blowup convex hull with few vertices.
  - Using the hull, we cover all integral points with polynomially many parallelepipeds.
  - If too many points are selected into λ_x, we redistribute their weight to the vertices of the parallelepiped.

The pre-baked ingredients

T(Solving integer programs of fixed dimension): Given $A \in \mathbb{Z}^{m \times d}$ and $b \in \mathbb{Z}^m$ with $\Delta \equiv \max(||A||_\infty, ||b||_\infty)$, one can find an $x \in \mathbb{Z}^d$ with $Ax \leq b$ (or deciding that none exists) in time $d^{\mathrm{O}(d)} \cdot m^{\mathrm{O}(1)}$.

T(Few vertices in an int. hull): Consider any polytope P with m constraints and $\Delta \equiv \max(||A||_\infty, ||b||_\infty) \geq 2$. Then $P_I = \mathrm{conv}(P \cap \mathbb{Z}^d)$ has at most $(m \cdot \mathrm{O}(\log \Delta))^d$ extreme points. In fact a list of the extreme points can be computed in time $d^{\mathrm{O}(d)} (m \cdot \mathrm{O}(\log \Delta))^{\mathrm{O}(d)}$.

T(Encapsulate a polytope by a blowup with few vertices): For a centrally symmetric polytope $P \subseteq \mathbb{R}^d$, there are $k \leq \frac{d}{2}(d+3)$ many extreme points $x_1, \ldots, x_k \in \mathrm{vert}(P)$ such that $P \subseteq \mathrm{conv}(\pm\sqrt{d} \cdot x_j | j \in [k])$.

T(Computing a minimum volume ellipsoid): Given a set of points S in $\mathbb{R}^d$, we can use SDP to compute a minimum volume ellipsoid E containing the given points in time polynomial to their encoding. Moreover, using the dual solution of the SDP, we can determine the contact points of $E \cap \mathrm{conv}(S)$.

Cooking

L(1): Let $P = \{x \in \mathbb{R}^d | Ax \leq b\}$ be a polytope defined by m inequalities with integral coefficients of absolute value at most Δ. Then there exists a set Par of at most $N \equiv m^d d^{\mathrm{O}(d)} (\log \Delta)^d$ integral parallelepipeds such that

$$P \cap \mathbb{Z}^d \subseteq \bigcup_{\Pi \in \mathrm{Par}} \Pi \subseteq P.$$

L(2): For any polytope $P \subseteq \mathbb{R}^d$ and any integral vector $\lambda \in \mathbb{Z}^{P \cap \mathbb{Z}^d}_{\geq 0}$ there exists a $\mu \in \mathbb{Z}^{P \cap \mathbb{Z}^d}_{\geq 0}$ such that $|\mathrm{supp}(\mu)| \leq 2^d$ and $\sum_x \mu_x x = \sum_x \lambda_x x$. Furthermore, $\mathrm{supp}(\mu) \subseteq \mathrm{conv}(\mathrm{supp}(\lambda))$.

L(3): Given an integral parallelepiped Π with vertices X. Then for any $x^* \in \Pi \cap \mathbb{Z}^d$ and $\lambda^* \in \mathbb{Z}_{d \geq 0}$ there is an integral vector $\mu \in \mathbb{Z}^{\Pi \cap \mathbb{Z}^d}_{\geq 0}$ such that the following holds:

1. $\lambda^* x^* = \sum_x \mu_x x$,
2. $|\mathrm{supp}(\mu \setminus X)| \leq 2^d$,
3. $\mu_x \in \{0, 1\} \forall x \notin X$.

P(Finding conic combinations): Let $P = \{x | Ax \leq b\}, Q = \{x | \overline{A} x \leq \overline{b}\}$.

Compute the set X of size at most $N = m^d d^{\mathrm{O}(d)} (\log \Delta)^d$ from the Structure Theorem in time $N^{\mathrm{O}(1)}$. Let y^* be the (unknown) target vector. Using the Structure Theorem, we get λ^*.

At the expense of a factor $N^{2^{2d}}$ guess $X' = X \cap \mathrm{supp}(\lambda^*)$. At the expense of factor $2^{2d} + 1$ guess the number $k = \sum_{x \notin X'} \lambda^*_x \in [2^{2d}]$ of extra points.

Create the following ILP:

$$\forall i \in [k] : Ax_i \leq b$$
$$\sum_{x \in X'} \lambda_x x + \sum_{i=1}^{k} x_i = y$$
$$\overline{A} y \leq \overline{b}$$
$$\forall x \in X' : \lambda_x \in \mathbb{Z}_{\geq 0}$$
$$\forall i \in [k] : x_i \in \mathbb{Z}^d$$

The number of variables is $X' + (k+1)d \leq 2^{\mathrm{O}(d)}$, the number of constraints is $km + d + \overline{m} + |X'|d = 2^{\mathrm{O}(d)} m + \overline{m}$. Maximal coefficient is $\max(d! \Delta^d, \overline{\Delta})$.

Bon appetit!