Klee's Measure Problem Made Easy

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Problem formulations

Given n axis-parallel d-dimensional boxes B (hyperrectangles) in $\mathbb{R}^d$ ...

Klee's measure problem
... determine the measure of their union $H^d(\bigcup B)$.

Maximum depth problem
... find a point $x \in \mathbb{R}^d$ that is contained in the maximum number of boxes.

Weighted maximum depth problem
... and weights $w : B \to \mathbb{R}$, find a point $x \in \mathbb{R}^d$ maximising $\sum_{x \in b \in B} w(b)$.

Coverage problem
... and an axis-parallel hyperrectangle Γ (the domain), does $\bigcup B$ cover Γ?

Small k-cluster
Given n points in $\mathbb{R}^d$ and a number k, find a subset of k points with minimal L_∞ diameter.

Graph k-clique
Given a graph on n nodes and a number k, is there a clique of size k?

Gradual improvements

J. L. Bentley, 1977: $O(n \log n)$ algorithm for measure in $\mathbb{R}^2$ (sweeping), $O(n^{d-1} \log n)$ for general d. Similarly for depth.

Overmars and Yap, FOCS 1988: $O(n^{d/2} \log n)$ algorithm for the measure problem. Similarly for depth.

T. M. Chan, 2010: $O(n^{d/2} 2^{\log^* n})$ algorithm for measure problem. Similarly for depth.

T. M. Chan, 2010: If the static d-dimensional measure (or coverage) problem can be solved in $T_d(n)$ time, then we can decide whether an arbitrary n-vertex graph contains a clique of size d in $O(T_d(O(n^2)))$ time.

The best combinatorial algorithms for k-clique currently runs in $O^*(n^k)$ (ignoring log-factors). The best algorithm using matrix multiplication runs roughly in $O(n^{wk/3})$ for $w \sim 2.376$.

Current results

T1: There is a simple $O(n^{d/2})$ algorithm for the measure problem.

T2: There is $O(n^{d/2}/\log^{d/2} n \log\log^{O(1)} n)$ algorithm for the depth and cover problem.

T3: There is $O(n^{d/2}/\log^{d/2-c} n \log\log^{O(1)} n)$, with constant $c < 5$, algorithm for the weighted depth problem.

T4: There is $O((n^{d/2}/\log^{d/2})/\log U \log\log^{O(1)} U)$ algorithm for the measure problem on word-RAM with integer coordinates $0 \dots U$.

T5: There is $O(n^{d/3} \log^{O(1)} n)$ algorithm for the measure problem of arbitrary orthants.

T6: There is $O(n^{(d+1)/3} \log^{O(1)} n)$ algorithm for the measure problem of arbitrary hypercubes.

Tools

L3.1: We can preprocess N linear functions $f_1, \dots, f_N : \mathbb{R}^b \to \mathbb{R}$ in time $(bN)^{O(b)}$ and then compute $f(x) = \max\{f_1(x), \dots, f_N(x)\}$ in time $O(b^c \log N)$ for any $x \in \mathbb{R}^b$ and $c \leq 5$.

L3.2: Given a polynomial $f : \mathbb{R}^b \to \mathbb{R}$ of degree $O(1)$ and $O(1)$ bounded integer coefficients, we can compute $S = \sum_{l=1}^{m} f(x^{(l)})$ for m b-tuples $x^{(1)}, \dots x^{(m)} \in [U]^b$ with all numbers from a set X, $|X| = n$, in time

$$O((m+n)\log U/\log\log U + mb\log b + 2^{O(b\log\log U)}).$$

Basic predicate $E(x_1, \dots x_d)$ is conjunction of $O(d^2)$ conditions of the form $x_j ? f_{i,j}(x_i)$, with $f_{i,j}$ monotone step function and ? either $\leq$ or $\geq$.

Basic function is of the form $F(x_1, \dots x_d) = [E(x_1, \dots x_d)] \cdot h_1(x_1) \cdot h_2(x_2) \dots h_d(x_d)$ with $h_i(x_i)$ piecewise-polynomial functions (density). Complexity of F is number of steps of $f_{i,j}$ and pieces of h_i.

L4.2: If F is basic of complexity n and degree s, then $F'(x_1, \dots x_d) = \int_{-\infty}^{x_d} F(x_1, \dots x_{d-1}, \xi) d\xi$ is a sum of $O(1)$ basic functions of complexity $O(n)$ and degree $s+1$, constructible in time $O(n+s)$.