Interlacing Families I: Bipartite Ramanujan Graphs of All Degrees

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- graph G, adjacency matrix A, if G is d-regular, d is always its eigenvalue, $-d$ is eigenvalue $\iff$ G is bipartite (*trivial* eigenvalues)
- *Ramanujan* graph - all non-trivial eigenvalues are in absolute value $\leq 2\sqrt{d-1}$

GOAL

- to construct an infinite family of d-regular Ramanujan graphs for all d
- this will be constructed as an infinite sequence of 2-lifts of Ramanujan graphs

COVERS

- 2-lift of $G = (V, E)$: $\bar{G} = (\bar{V}, \bar{E})$ $\bar{V} = \{u_1, u_2 \forall u \in V\}$ and

$$\forall (u, v) \in E \begin{cases} (u_1, v_1), (u_2, v_2) \in \bar{E} \\ (u_1, v_2), (u_2, v_1) \in \bar{E} \end{cases}$$

- corresponding *signing* s of the edges by ± 1, corresponding signed adjacency matrix A_s
- eigenvalues of a two lift are the union of eigenvalues of G and the eigenvalues of A_s
- *universal cover* of a graph G is an infinite tree such that every connected lift of G is a quotient of the tree
- *path-tree* of a graph G, u ($u \in V(G)$) contains one vertex for every non-backtracking path in G that starts in u
- every path-tree of G is an induced subgraph of the universal cover of G
- eigenvalues of a d-regular universal cover are $|\lambda| \leq 2\sqrt{d-1}$ ((c, d)-biregular, then $|\lambda| \leq \sqrt{d-1} + \sqrt{c-1}$)

ROOTS OF THE EXPECTED VALUE OF THE CHAR. POLY. OF A_s ARE $\leq 2\sqrt{d-1}$

- *matching polynomial* of G is

$$\mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i$$

  where $m_0 = 1$ and m_i is the number of matchings in G with i edges for $i > 0$

- *spectral radius* of graph G is $\rho(G) = \max\{\|Ax\|_2, \|x\|_2 = 1\}$, where λ_i are the eigenvalues of its adjacency matrix A ($\rho(G) = \sup\{\|Ax\|_2, \|x\|_2 = 1\}$ for A infinite-dimensional)

Theorem. 3.1. *For every graph G, μ_G has only real roots.*

Theorem. 3.2. *For every graph G of maximum degree d, all roots of μ_G have absolute value at most $2\sqrt{d-1}$.*

Theorem. 3.4. *Let $T(G, u)$ be a path-tree, then μ_G divides the characteristic polynomial of the adjacency matrix of $T(G, u)$, i.e. all roots of μ_G are real with absolute value at most $\rho(T(G, u))$.*

Theorem. 3.5. *Let G be a graph and T its universal cover. Then the roots of the matching polynomial of G are bounded in absolute value by the the spectral radius of T.*

Theorem. 3.6. $\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)] = \mu_G(x)$.

INTERLACING POLYNOMIALS - USEFUL ROOT BOUNDARIES

- $g(x) = \prod_{i=1}^{n-1}(x-\alpha_i)$, $f(x) = \prod_{i=1}^{n}(x-\beta_i)$, g *interlaces* f if: $\beta_a \leq \alpha_1 \leq \beta_2 \leq \ldots \leq \alpha_{n-1} \leq \beta_n$

Theorem. 4.2. *Let $f_1, \ldots f_k$ be real-rooted polynomials of the same degree, with positive leading coefficient, $f_\emptyset = \sum_{i=1}^{k} f_i$. If $f_1, f_2, \ldots f_k$ have a common interlacing, then there exists an i such that the largest root of f_i is at most the largest root of $f_\emptyset$.*

- $S_1, \ldots S_m$ finite sets and for every $s_1 \in S_1, \ldots s_m \in S_m$ let $f_{s_1,\ldots s_m}$ be a real-rooted degree n polynomial with positive leading coefficients.
  For every partial assignment $s_1 \in S_1, \ldots s_k \in S_k$ define

$$f_{s_1,\ldots s_k} = \sum_{s_{k+1}\in S_{k+1},\ldots s_m \in S_m} f_{s_1,\ldots s_k, s_{k+1}\ldots s_m}$$

$$f_\emptyset = \sum_{s_1\in S_1,\ldots s_m\in S_m} f_{s_1,\ldots s_m}.$$

- if for all $k = 0, 1, \ldots m-1$ and all $s_1 \in S_1 \ldots s_k \in S_k$ the polynomials $\{f_{s_1,\ldots s_k,t}\}_{t\in S_{k+1}}$ have a common interlacing, then $\{f_{s_1,\ldots s_m}\}_{s_1,\ldots s_m}$ form an *interlacing family.*

Theorem. 4.4. *Let $S_1, \ldots S_m$ be finite sets, and let $\{f_{s_1,\ldots s_m}\}$ be an interlacing family of polynomials. Then there exists some $s_1, \ldots s_m \in S_1 \times \ldots \times S_m$ so that the largest root of $\{f_{s_1,\ldots s_m}\}$ is less than the largest root of $f_\emptyset$.*

Theorem. 4.5. *Let f and g be (univariate) polynomials of degree n such that for all $\alpha, \beta > 0$, $\alpha f + \beta g$ has n real roots. Then f and g have a common interlacing.*

SIGNED CHAR. POLY. ARE $\mathbb{R}$-ROOTED AND FORM INTERLACING FAMILY

Theorem. 5.1. *Let $p_1, \ldots p_m$ be be numbers in $[0,1]$. Then the following polynomial is real-rooted:*

$$\sum_{s\in\{\pm1\}^m} (\prod_{i:s_i=1} p_i)(\prod_{i:s_i=-1})f_s(x).$$

Theorem. 5.2. *The polynomials $\{f_s\}_{s\in\{\pm1\}^m}$ are an interlacing family.*

WE ARE ALMOST DONE :)

Theorem. 5.3. *Let G be a graph with adjacency matrix A and universal cover T. Then there is a signing s of A such that all of the eigenvalues of A_s are at most $\rho(T)$, i.e. for d-regular graphs, the eigenvalues of A_s are at most $2\sqrt{d-1}$*

Theorem. 5.4. *For every $d \geq 3$ there is an infinite sequence of d-regular bipartite Ramanujan graphs.*

Theorem. 5.5. *For every $c, d \geq 3$ there is an infinite sequence of (c,d)-biregular bipartite Ramanujan graphs, with nontrivial eigenvalues bounded by $\sqrt{c-1} + \sqrt{d-1}$.*

SOME MORE DEFINITIONS & PROOF OF THM 5.1.

- multivariate polynomial $f \in \mathbb{R}[z_1, \ldots z_n]$ is *real stable* if $f(z_1, \ldots z_n) \neq 0$ whenever the imaginary part of every z_i is strictly positive.

Theorem. 6.2. *Let $f(z_1, \ldots z_n) + \omega g(z_1, \ldots z_n) \in \mathbb{R}[z_1, \ldots z_n, \omega]$ be a real stable of degree at most 1 in z_j. Then the following polynomial will also be real stable:*

$$f(z_1, \ldots z_n) - \frac{\partial g}{\partial z_j}(z_1, \ldots z_n).$$

Theorem. 6.3. *For any real stable polynomials $f(z_1, \ldots z_n)$ and $t(\omega_1, \ldots \omega_m)$ with $m \leq n$ which both have degree at most 1 in the variables z_j, ω_j for $a \leq j \leq m$, the polynomial below will also be real stable:*

$$t(-\frac{\partial g}{\partial z_1}, \ldots, -\frac{\partial g}{\partial z_m})f(z_1, \ldots z_n).$$

Theorem. 6.4. *Let $A_1, \ldots A_m$ be positive semidefinite matrices. Then $det[z_1A_1 + \ldots + z_mA_m]$ is a real stable polynomial.*

Theorem. 6.5. *Let $a_1, \ldots a_m$ and $b_1, \ldots b_m$ be vectors in $\mathbb{R}^n$ and let $p_1, \ldots p_m$ be real numbers in $[0,1]$. Then every (univariate) polynomial of the form below is real stable:*

$$P(x) = \sum_{S\subseteq[m]} (\prod_{i\in S} p_i)(\prod_{i\notin S}(1-p_i))det[xI + \sum_{i\in S} a_i a_i^T + \sum_{i\notin S} b_i b_i^T].$$