Convex equipartitions: The spicy chicken theorem

Roman Karasev, Alfredo Hubard, Boris Aronov

Presented by Zuzana Safernová

Conjecture 1. *(Nandakumar and Ramana Rao) Can a convex body in the plane be partitioned into n convex regions with equal areas and equal perimeters?*

Corollary 2. *Given a convex body K in $\mathbb{R}^d$, and a prime power n, it is possible to partition K into n convex bodies with equal d-dim volumes and equal $(d-1)$-dim surface areas.*

- *Absolutely continuous (a.c.) measure* μ: $\lambda(A) = 0 \Rightarrow \mu(A) = 0$
- $\mathcal{K}^d$ – space of convex sets in $\mathbb{R}^d$ with Hausdorff metric

Theorem 3. *Given an a.c. finite measure μ on $\mathbb{R}^d$, a convex body $K \in \mathcal{K}^d$, a family of $d-1$ continuous functionals $\varphi_1, \varphi_2, \ldots, \varphi_{d-1} \colon \mathcal{K}^d \to \mathbb{R}$, and a prime power n, there is a partition of K into n convex bodies $K_1, K_2, \ldots, K_n$, such that $\mu(K_i) = \frac{\mu(K)}{n}$ and $\varphi_j(K_1) = \varphi_j(K_2) = \cdots = \varphi_j(K_n)$, for all $1 \le i \le n$ and $1 \le j \le d-1$.*

Corollary 4. *Given d a.c. probability measures $\mu_1, \ldots, \mu_d$ on R^d, and any number n, there is a partition of $\mathbb{R}^d$ into convex regions $K_1, \ldots, K_n$ with $\mu_i(K_j) = \frac{1}{n}$ for all i, j simultaneously.*

Theorem 5. *Suppose $K \in \mathcal{K}^d$ is a convex body, and, for some $1 \le m \le d$, we have m a.c. finite measures $\mu_1, \ldots, \mu_m$ on K, and $d-m$ a.c. finite measures $\sigma_1, \ldots, \sigma_{d-m}$ on ∂K. Then, for any n, the body K can be partitioned into n convex parts $K_1, \ldots, K_n$, such that,*

- *for any $i = 1, \ldots, m$ we have $\mu_i(K_1) = \cdots = \mu_i(K_n)$, and*
- *for every $i = 1, \ldots, d-m$ we have $\sigma_i(K_1 \cap \partial K) = \cdots = \sigma_i(K_n \cap \partial K)$.*

Theorem 6. *Given a convex body $K \in \mathcal{K}^d$, an a.c. finite measure μ on K, a prime power n, a continuous map $g \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$, and a continuous centermap $c \colon \mathcal{K}^d \to \mathbb{R}^d$, then there exists a partition of K into n convex sets $K_1, \ldots, K_n$, such that $\mu(K_i) = \frac{\mu(K)}{n}$, for all i, and $g(c(K_1)) = \cdots = g(c(K_n))$.*

- *Configuration space* $F_n(\mathbb{R}^d) := \{(x_1, \ldots, x_n) \in \mathbb{R}^{nd} \colon x_i \ne x_j \text{ for all } i \ne j\}$
- Symmetric group Σ_n acts on $F_n(\mathbb{R}^d)$ by permuting the points in a tuple and on $\mathbb{R}^n$ by permuting the coordinate axes.
- Denote by α_n the orthogonal complement of the diagonal. Restrict the action of Σ_n on $\mathbb{R}_n$ to the action on α_n.
- A map $f \colon \mathbb{R}^n \to \mathbb{R}^m$, $n \ge m$, is Σ_n-equivariant if $f \circ \pi = \pi \circ f$, for $\pi \in \Sigma_n$.

Theorem 7. *(Fuchs, Vasiliev, Karasev) Let n be a prime power. For any Σ_n-equivariant continuous map $f \colon F_n(\mathbb{R}^d) \to \alpha_n^{\oplus(d-1)}$, there exists a configuration $\bar{x} \in F_n(\mathbb{R}^d)$, such that $f(\bar{x}) = 0$.*

- X – a compact top. space with a Borel probabilistic measure μ
  $\mathcal{C}(X)$ – a set of real-valued continuous functions on X
- A fin. dim. linear subspace $L \subset \mathcal{C}(X)$ is *measure separating* (m.s.) if, for any $f \ne g \in L$, the measure of the set $e(f, g) = \{x \in X \colon f(x) = g(x)\}$ is zero.
- Let $F = \{u_1, \ldots, u_n\} \subset \mathcal{C}(X)$ and $\mu(e(u_i, u_j)) = 0$. The sets $V_i = \{x \in X \colon \forall j \ne i, u_i(x) \le u_j(x)\}$ define a partition $P(F)$ of X.

Theorem 8. *Suppose L is a m.s. subspace of $\mathcal{C}(X)$ of dimension $d+1$, $\mu_1, \ldots, \mu_d$ are a.c. probability measures on X. Then, for any prime power n, there exists a subset $F \subset L$, $|F| = n$ such that, for every $i = 1, \ldots, d$, the family $P(F)$ partitions the measure μ_i into n equal parts.*