2-Cancellative Hypergraphs and Codes
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1 Definitions and notation

Definition 1. $\mathcal{F}$ a family of sets is t-cancellative if for all $t+2$ sets $A_1, \ldots A_t, B, C \in \mathcal{F}$ such that $A_i \neq B$, $A_i \neq B$

$$A_1 \cup A_2 \cup \ldots \cup A_t \cup B = A_1 \cup A_2 \cup \ldots \cup A_t \cup C \to B = C.$$

Cancellative means 1-cancellative.

- $c(n)$ – the size of the largest cancellative family on n elements
- $c(n,r)$ – the size of the largest r-uniform cancellative family on n elements
- $f(n, P_1, P_2, \ldots)$ – the maximum number of subsets of $\{1, 2, \ldots n\}$ satisfying properties $P_1 P_2, \ldots$

Observation 1. $c(n+m) \geq c(n)c(m)$

Definition 2. Hypergraph $\mathbb{F} = (V, \mathcal{F})$ is (k) uniform is each edge has the same number of elements, and linear if $|E \cap F| \leq 1$ for all edges $E, F \in \mathcal{F}$.

Definition 3. Associate each subset of $\mathcal{F}$ to its characteristic vector. If it satisfies, that for each a-tuple of these vectors at least b different columns sum up to 1, it is locally (a,b)-thin.

Definition 4. $\mathcal{F} \subseteq 2^n$ is g-cover-free if it is locally $(g+1, g+1$-thin, i.e. it suffices $A_0 \not\subseteq \bigcup_{i=1}^{g} A_i$ for all $A_0, A_1, \ldots A_g \in \mathcal{F}$.

- $C_g(n)$ – the size of the largest g-cover-free n vertex code
- $C_g(n,r)$ – the size of the largest g-cover-free r-uniform hypergraph on n vertices

Observation 2.
- $C_g(n) \leq C_{g-1}(n) \leq \ldots \leq C_1(n)$
- $C_g(n,r) \leq C_{g-1}(n,r) \leq \ldots \leq C_2(n,r)$
- $C_{t+1}(n,r) \leq c_t(n,r)$ because $t+1$-cover-free $\iff$ t-cancellative

2 Estimates

Lemma 1. (D'yachkov, Rykov) $\exists \alpha_1, \alpha_2$ such that $\alpha_1 \frac{1}{g^2} < \frac{\log(C_g(n))}{n} < \alpha_2 \frac{\log(g)}{g^2}$

Theorem 1. *$\exists \beta_1, \beta_2$ and $n_0(t)$ such that $\forall n > n_0(t), t \geq 2$ holds $\beta_1 \frac{1}{t^2} < \frac{\log(c_t(n))}{n} < \beta_2 \frac{\log(t)}{t^2}$*

Theorem 2. $\forall n, k \in \mathbb{Z}$: $c_2(n, 2k) \leq \frac{\binom{n}{k}}{\frac{1}{2}\binom{2k}{k}}$

Theorem 3. $c_t(n) < \alpha n^{\frac{t-1}{2}} (\frac{t+3}{t+2})^n$

Lemma 2. $\mathcal{F}$ an r-uniform hypergraph, then $\exists \mathcal{F}^* \subseteq \mathcal{F}$ such that $\mathcal{F}^*$ is r-partite and $|\mathcal{F}^*| \geq \frac{r!}{r^r}\ |\mathcal{F}|$

Theorem 4. $f_3(n, 7, 4) - \frac{2}{5}n \leq c_2(n, 3) \leq \frac{9}{2}f_3(n, 7, 4) + n$

Lemma 3. (F., Frankl) $i(n, H) = \frac{1}{e(H)}\binom{n}{2} - o(n^2)$ where $i(n, H)$ is the maximal number of almost disjoint induced copies of H that can be packed into any n-vertex graph.

Theorem 5. $c_2(n, 4) = \frac{1}{6}n^2 - o(n^2)$

Definition 5. $\mathcal{P} = \{G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots\}$ is a packing if all the graphs are edge-disjoint subgraphs of some $G = (V, \mathcal{E})$.
An induced packing is calles almost dijoint induced packing into G, if $|V_i \cap V_j| \leq 2$ for all $i \neq j$, i.e. any induced $G[V_i]$, $G[V_j]$ are vertex disjoint, share 1 vertex, or the set of their intersection is not a subset of any edge of G.

Theorem 6. $c_2(n, 2k) \geq \frac{n^k}{(2k)^k} - o(n^k)$

Definition 6. Symmetric polynomial is defined as $\sigma_i(X) = \sum_{I \subseteq I, |I|=i} \prod_{\alpha \in I} x_\alpha$, where $X = \{x_1, x_2, \ldots x_s\}, 0 \leq i \leq s$ and $\sigma_0(X) = 1$.
5 $X_1, \ldots X_l$ disjoint, $|X_j| = t_j, 0 < t_j < k, \sum_j (k - t_j) = k$, matrix $M(X_1, \ldots X_l)\ldots$

Observation 3.

- $det(M(X_1, \ldots X_l))$ is non-vanishing

- $\mathbf{F_q^s}$ a field, q power of a prime, $Z(p) = \{(x_1, \ldots x_s) \in \mathbf{F_q^s} : p(x_1, \ldots x_s) = 0\}$
- $\mathcal{P}_{\ <k} = \{a_0 + a_1 x + \ldots + a_{k-1}x^{k-1} : a_i \in F\}$
- $p_Z(x) = \prod_{z \in Z}(x - z)$

Definition 7. $p_1(x), \ldots p_l(x)$ are $(k_1, \ldots k_l)$-independent if
$f_1(x)p_1(x) + \ldots + f_l(x)p_l(x) = 0, deg(f_i) < k_i \rightarrow f_i(x) = 0$ for all i.

- let $l \geq 2$, $k_i \in \mathbb{Z}$, $\sum k_i = k$, $x_i \in \mathbf{F_q}$, $1 \leq i \leq (l-1)k$, then multiset

$$X_1 = \{x_s : 1 \leq s \leq k - k_1\}, \qquad X_j = \{x_s : \sum_{i<j}(k - k_i) < s \leq \sum_{i \leq j}(k - k_i)\}$$

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Lemma 4. Polynomials $p_{X_1}(x), \ldots p_{X_l}(x)$ are $(k_1, \ldots k_l)$-independent for all but at most $\binom{lk}{2}q^{(l-1)k-1}$ sequences.

Lemma 5. $\forall k \exists q_0(k)$: if $q > q_0(k)$ then $\exists S \subseteq \mathbf{F_q}$, $|S| = 2k$ such that polynomials

$$p_X(x), p_Y(x), p_W(x)$$

are $(k - |X|, k - |Y|, k - |W|)$-independent for each partition

$$S = X \cup Y \cup W,$$

$1 \leq |X|, |Y|, |W| \leq k$, $|X| + |Y| + |W| = 2k$.

Theorem 7. $\forall n \geq r \geq 2$ *holds:* $c(n, r) > \frac{\gamma_0}{2^r}\binom{n}{r}$, *where* $\gamma_0 = \prod_{k \geq 1} \frac{2^k - 1}{2^k} = 0,2887\ldots$