The critical window for the classical Ramsey-Turán problem

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For a graph H and positive integers n and m, the Ramsey-Turán number $\mathbf{RT}(n, H, m)$ is the maximum number of edges a graph G on n vertices with independence number less than m can have without containing H as a subgraph. The first application of Szemerédi's powerful regularity method was the following celebrated Ramsey-Turán result.

Theorem 1 (Szemerédi, 1972)**.** *For every $\epsilon > 0$, there is a $\delta > 0$ for which every n-vertex graph with at least $(\frac{1}{8} + \epsilon)n^2$ edges contains either a K_4 or an independent set larger than δn.*

Later, Bollobás and Erdős gave a geometric construction of a K_4-free graph on n vertices with independence number $o(n)$ and $(\frac{1}{8} - o(1))n^2$ edges. Since then, several problems have been asked on estimating the minimum possible independence number in the critical window, when the number of edges is about $\frac{n^2}{8}$. In this paper, the authors give nearly best-possible bounds, solving the various open problems concerning the critical window.

The first result is a new proof of Theorem 1 which gives a much better bound and completely avoids using the regularity lemma or any notion similar to regularity.

Theorem 2. *For every α and n, every n-vertex graph with at least $\frac{n^2}{8} + 10^{10}\alpha n$ edges contains either a copy of K_4 or an independent set of size greater than α.*

The second result sharpens the linear dependence down to a very reasonable constant.

Theorem 3. *There is an absolute positive constant γ_0 such that for every $\alpha < \gamma_0 n$ every n-vertex graph with at least $\frac{n^2}{8} + \frac{3}{2}\alpha n$ edges contains a copy of K_4 or an independent set of size greater than α.*

The authors also prove the following corresponding lower bound, which shows that the linear dependence in Theorem 3 is best possible.

Theorem 4. *For $\frac{(\log\log n)^{3/2}}{(\log n)^{1/2}} \cdot n \ll m \leq \frac{n}{3}$, we have $\mathbf{RT}(n, K_4, m) \geq \frac{n^2}{8} + \big(\frac{1}{3} - o(1)\big)mn$.*

By introducing a new twist on the dependent random choice technique, the authors substantially improve the lower bound on the independence number at the critical point of exactly $\frac{n^2}{8}$ edges.

Theorem 5. *There is an absolute positive constant c such that every n-vertex graph with at least $\frac{n^2}{8}$ edges contains a copy of K_4 or an independent set of size greater than $cn \cdot \frac{\log\log n}{\log n}$.*

Modifying the Bollobás-Erdős graph, we also get an upper bound on this problem.

Theorem 6. *There is an absolute positive constant c' such that for each positive integer n, there is an n-vertex K_4-free graph with at least $\frac{n^2}{8}$ edges and independence number at most $c'n \cdot \frac{(\log\log n)^{3/2}}{(\log n)^{1/2}}$.*

The following result shows that the Bollobás-Erdős graph gives a good lower bound for the Ramsey-Turán numbers in the lower part of the critical window, nearly matching the upper bounds established using dependent random choice.

Theorem 7. *If $m = e^{-o((\log n/\log\log n)^{1/2})}n$, then $\mathbf{RT}(n, K_4, m) \geq (1/8 - o(1))n^2$.*

1 The new proof of Theorem 1

The standard regularity proof

Definitions:

- The *edge density* $d(X, Y)$ between two subsets of vertices of a graph G is the fraction of pairs $(x, y) \in X \times Y$ that are edges of G.
- A pair (X, Y) of vertex sets is called *ϵ-regular* if for all $X' \subset X$ and $Y' \subset Y$ with $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$, we have $|d(X', Y') - d(X, Y)| < \epsilon$.
- A partition $V = V_1 \cup \ldots \cup V_t$ is called *equitable* if $||V_i| - |V_j|| \leq 1$ for all i and j.

The regularity lemma states that for each $\epsilon > 0$, there is a positive integer $M(\epsilon)$ such that the vertices of any graph G can be equitably partitioned $V = V_1 \cup \ldots \cup V_t$ into $\frac{1}{\epsilon} \leq t \leq M(\epsilon)$ parts where all but at most ϵt^2 of the pairs (V_i, V_j) are ϵ-regular.

The standard regularity proof

Lemma 1. *Let $t, \gamma > 0$ satisfy $\gamma t \le 1$. If G is a K_4-free graph on n vertices with independence number at most γn, and X and Y are disjoint vertex subsets of size n/t, then the edge density between X and Y is at most $\frac{1}{2} + \gamma t$.*

Lemma 2. *Suppose that X, Y, and Z are disjoint subsets of size m, and each of the three pairs are ϵ-regular with edge density at least 3ϵ. Then there is either a K_4 or an independent set of size at least $4\epsilon^2 m$.*

The new proof

Theorem 8 (Erdős, Simonovits)**.** *For every $\epsilon > 0$, there is a $\delta > 0$ such that every n-vertex triangle-free graph with more than $\left(\frac{1}{4} - \delta\right) n^2$ edges is within edit distance ϵn^2 from a complete bipartite graph.*

Lemma 3. *For every $c > 0$ there is a $\gamma > 0$ such that every K_4-free graph G on n vertices with at least $\frac{n^2}{8}$ edges and independence number at most γn has a cut which has at most cn^2 non-crossing edges.*

Lemma 4. *Let G be an n-vertex graph with at least m edges. Then G contains an induced subgraph G' with $n' > 2m/n$ vertices, at least $n' \frac{m}{n}$ edges, and minimum degree at least $\frac{m}{n}$.*

Lemma 5. *Let G be a K_4-free graph on n vertices, at least $\frac{n^2}{8}$ edges, and independence number at most $\alpha \le cn$. Suppose its vertices have been partitioned into $L \cup R$, and $e(L) + e(R) \le cn^2$. Then $|L|$ and $|R|$ are both within the range $\left(\frac{1}{2} \pm \sqrt{3c}\right) n$.*

Lemma 6. *Let G be a K_4-free graph on n vertices with minimum degree at least cn and independence number at most $\alpha \le \frac{cn}{36}$. Let $L \cup R$ be a max-cut with $\frac{n}{3} \le |R| \le \frac{2n}{3}$. Let $T \subset L$ be the vertices with R-degree greater than $\left(\frac{1}{2} - \frac{c}{8}\right) |R|$. Then every vertex of L has at most α neighbors in T.*

Lemma 7. *For any $0 < c < 1$, the following holds with $c' = c^2/800$. Assume $\alpha < cn/300$. Let G be a K_4-free graph on n vertices with at least $\frac{n^2}{8} + \frac{3\alpha n}{2}$ edges, and minimum degree at least cn. Suppose that the max-cut of G partitions the vertex set into $L \cup R$ such that $e(L) + e(R) \le c'n^2$. Then G either has a copy of K_4, or an independent set of size greater than α.*

2 Quantitative bounds on the Bollobás-Erdős construction

Call a graph $G = (V, E)$ on n vertices *nice* if it is K_4-free and there is a bipartition $V = X \cup Y$ into parts of order $n/2$ such that each part is K_3-free.

Theorem 9. *There exists some universal constant $C > 0$ such that for every $0 < \epsilon < 1$, positive integer $h \ge 16$ and even integer $n \ge \left(C\sqrt{h}/\epsilon\right)^h$, there exists a nice graph on n vertices, with independence number at most $2ne^{-\epsilon\sqrt{h}/4}$, and minimum degree at least $(1/4 - 2\epsilon)n$.*

Theorem 10 (Schmidt, 1948)**.** *Let $l \in [0, 2]$ and h be a positive integer. If $A \subset \mathbb{S}^{h-1}$ is an arbitrary measurable set with diameter at most l and B a spherical cap in $\mathbb{S}^{h-1}$ with diameter l, then $\lambda(A) \le \lambda(B)$ (where λ is the Lebesgue measure).*

Corollary 1 (Simonovits, Sós, 2001)**.** *Let $\mu \in [0, 1)$. If $A \subset \mathbb{S}^{h-1}$ is any measurable set with diameter at most $2 - \mu$, then $\lambda(A) \le 2e^{-\mu h/2}$ (here λ is normalized to 1 on $\mathbb{S}^{h-1}$)*

Lemma 8. *Let $h \ge 5$ be positive integer, and $\epsilon > 0$. Let B be the spherical cap in $\mathbb{S}^{h-1}$ consisting of all points with distance at most $\sqrt{2} - \frac{\epsilon}{\sqrt{h}}$ from some fixed point. Then $\lambda(B) \ge \frac{1}{2} - \sqrt{2}\epsilon$.*

Let $S(m, n)$ be the maximum number of edges of a nice graph on n vertices with independence number less than m. Note that $\mathbf{RT}(n, K_4, m) \ge S(m, n)$.

Corollary 2. *For n sufficiently large and $\delta = 4(\log\log n)^{3/2}/(\log n)^{1/2}$, we have $\mathbf{RT}(n, K_4, \delta n) \ge S(n, \delta n) \ge (1/8 - \delta)n^2$.*

Lemma 9. *For positive integers d, m, n with $n \ge 6$ even and $d \le n/2$, we have $S(n, m+d) \ge \left(1 - \frac{2d}{n}\right)^2 S(n, m) + dn - d^2 - n$.*

Corollary 3. *For even $n \ge 6$, if $S(n, m) \ge (\frac{1}{8} - \delta)n^2$ with $n^{-1/2} \le \delta \le \frac{1}{4}$, then $S(n, m + 2\delta n) \ge \frac{n^2}{8}$.*

Corollary 4. *For even $n \ge 6$, if $S(n, m) \ge (\frac{1}{8} - \delta)n^2$ and $\frac{1}{\delta n} \le a \le \frac{1}{2}$, then $S(n, m + an) \ge \frac{n^2}{8}(1 + 4a - 4a^2 - 8\delta)$.*