On the Caccetta-Häggkvist Conjecture with Forbidden Subgraphs

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Conjecture 1 (Caccetta-Häggkvist Conjecture) Any $\vec{C}_3$-free orgraph Γ on n vertices contains a vertex v with $d^+_\Gamma(v) \leq \frac{n-1}{3}$.

1. Extremal configurations

Define the (infinite) orgraph Γ_0 with $V(\Gamma_0) \stackrel{\rm def}{=} S^1$ (unit circle) and $E(\Gamma_0) \stackrel{\rm def}{=} \{\langle x,y\rangle \mid y-x<1/3 \mod 1\}$. Let $\Omega \stackrel{\rm def}{=} (S^1)^\infty$ be the infinite-dimensional torus. Define the orgraph $\Gamma_{\rm CH}$ with $V(\Gamma_{\rm CH}) = \Omega$ and for any two different vertices $\mathsf{x} = (x_1, x_2, \ldots, x_n, \ldots)$, $\mathsf{y} = (y_1, \ldots, y_n \ldots) \in \Omega$ we choose the minimal d for which $x_d \neq y_d$ and let $\langle \mathsf{x}, \mathsf{y}\rangle \in \mathsf{E}(\Gamma_{\rm CH})$ if and only if $\langle x_d, y_d\rangle \in E(\Gamma_0)$.

Fix a probability measure μ on Borel subsets of Ω. Every finite string $(a^1, \ldots, a^d) \in (S^1)^d$ defines the canonical closed set $\Omega_a = \{\mathsf{x} \in \Omega \mid x_1 = a_1, \ldots, x_d = a_d\}$. Whenever $\mu(\Omega_a) > 0$, we have the conditional measure μ_a on Ω_a ($\mu_a(X) \stackrel{\rm def}{=} \frac{\mu(X)}{\mu(\Omega_a)}$, $X \subseteq \Omega_a$) and then the pushforward measure $\widehat{\mu}_a$ on S^1 defined by projecting Ω_a onto the $(d+1)$st coordinate. Let us call the measure μ *extremal* if for every prefix a for which $\mu(\Omega_a) > 0$, this measure $\widehat{\mu}_a$ has one of the following two forms:

- uniform (Lebesgue) measure on S^1;
- uniform discrete measure on the set $\left\{\frac{0}{3h+1}, \frac{1}{3h+1}, \ldots, \frac{3h}{3h+1}\right\}$ for some integer $h \geq 1$.

Claim 1.1 *For any extremal measure μ on Ω with the above property and for any $\mathsf{x} \in \Omega$,*

$$\mu(\{\mathsf{y} \in \Omega \mid \langle \mathsf{x}, \mathsf{y}\rangle \in \mathsf{E}(\Gamma_{CH})\}) = 1/3.$$

2. Main theorem

Theorem 2.1 *Let Γ be an orgraph on n vertices that does not contain either $\vec{C}_3$ or any of the three orgraphs on Figure 1 as an induced subgraph. Then Γ contains a vertex v with $d^+_\Gamma(v) \leq \frac{n-1}{3}$.*

Definition 2.2 (Types and flags) 0 and 1 are the unique type of sizes 0 and 1, respectively.

A is the type of size 2 with $E(A) \stackrel{\rm def}{=} \{\langle 1,2\rangle\}$, and N is the type of size 2 without any edges. P is the type of size 3 with $E(P) \stackrel{\rm def}{=} \{\langle 1,2\rangle, \langle 2,3\rangle\}$.

$\alpha \in \mathcal{F}^1_2$ is a directed edge in which the tail vertex is labeled by 1.

Our final goal is to find a vertex v with $\alpha(v) \leq 1/3$.

For a type σ of size k, let $O^\sigma \in \mathcal{F}^\sigma_{k+1}$ be the flag in which the only free vertex has k incoming edges.

Let us call an edge $\langle v, w\rangle$ *critical* if $O^A(v,w)$ takes the minimal possible value over all edges going out of v.

Claim 2.3 *For any critical edge $\langle v, w\rangle$, $\widehat{O}^A(v,w) = 0$.*

In what follows, we argue by contradiction, i.e. we assume that $\alpha(v) > \frac{1}{3}$ for all $v \in V(\Gamma)$.

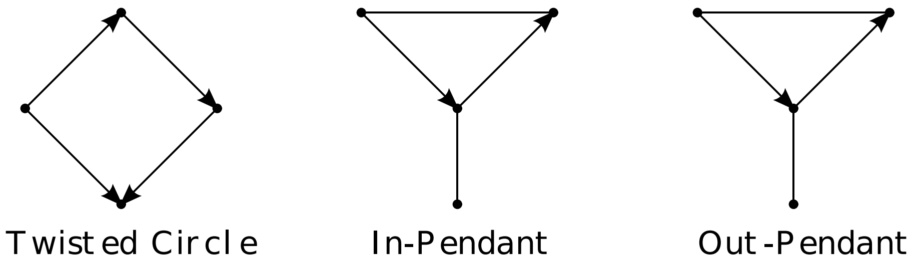

Figure 1: Forbidden orgraphs.

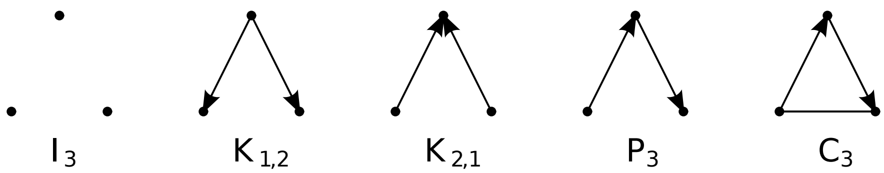

Figure 2: Some orgraphs on 3 vertices.

Claim 2.4 *For any critical edge $\langle v, w\rangle$, $\vec{P}_3^A(v,w) > 0$.*

Claim 2.5 *If $\langle u, v\rangle$ and $\langle v, w\rangle$ are critical edges then u and w are independent.*

Claim 2.6 *If $\langle u, v\rangle$ and $\langle v, w\rangle$ are critical edges then $\vec{K}_{2,1}^N(u,w) = 0$.*

Claim 2.7 *If $\langle u, v\rangle$ and $\langle v, w\rangle$ are critical edges then*

$$3O^A(u,v) \leq \vec{P}_3^N(u,w) - \frac{1}{n-2}. \tag{1}$$

Claim 2.8 *If $\langle u, v\rangle$ and $\langle v, w\rangle$ are critical edges, then*

$$\left.\begin{array}{l} \alpha(u) + \alpha(v) + \alpha(w) + (O^A(u,v) + I^A(u,v) + \vec{K}_{2,1}^A(u,v)) \\ \quad -(O^A(v,w) + I^A(v,w) + \vec{K}_{2,1}^A(v,w)) \leq 1. \end{array}\right\} \tag{2}$$

Let us pick $\boldsymbol{x} \in V(\Gamma) \setminus \{u, v, w\}$ uniformly at random and let us re-calculate all quantities in the left-hand side with respect to that distribution. Denoting these re-calculated quantities with $\widetilde{\alpha}(u), \ldots, \widetilde{\vec{P}_3^N}(u,w)$, we claim that

$$\left.\begin{array}{l} \widetilde{\alpha}(u) + \widetilde{\alpha}(v) + \widetilde{\alpha}(w) + (\widetilde{I^A}(u,v) + \widetilde{\vec{K}_{2,1}^A}(u,v) - 2\widetilde{O^A}(u,v)) \\ \quad -(\widetilde{O^A}(v,w) + \widetilde{I^A}(v,w) + \widetilde{\vec{K}_{2,1}^A}(v,w)) + \widetilde{\vec{P}_3^N}(u,w) \leq 1. \end{array}\right\} \tag{3}$$

For that we prove that every *individual* $x \notin \{u, v, w\}$ contributes at most 1 to the left-hand side.