Hypercontractivity, Sum-of-Squares Proofs, and their Applications

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Conj(*UGC*): For every $\varepsilon > 0$, the following problem is NP-hard:

"Given a system of equations $x_i - x_j = c \bmod k$, answer **Yes** at least $1 - \varepsilon$ of equations are satisfiable, **No** otherwise."

UGC implies that for a large class of problems (**Max Cut, Vertex Cover, Max CSP**) SDP-approximations are the best possible.

D: $\Phi_G(S) = \frac{E(S,V-S)}{d|S|}$ and $\Phi_G(\delta)$ is the minimum of $\Phi_G(S)$ over sets with relative size δ.

Conj(*small-set expansion*): For every $\eta > 0$, there exists $\delta > 0$ such that the following problem is NP-hard:

"Given a (regular) graph G, answer **Yes** if $\Phi_G(\delta) \geq 1 - \eta$ and **No** otherwise."

Claim: SSEH implies UGC, the converse is not yet known.

Two main results of this work:

- An algorithm that solves all known hard UGC instances, including ones hard for other algorithms → UGC might not hold.
- SSEH, a natural strengthening of UGC, needs quasi-polynomial time → UGC might hold.

D: A $p \to q$ norm $||A||_{p\to q}$ of a linear operator A between vector spaces of functions $\Omega \to \mathbb{R}$ is the smallest number $c \geq 0$ such that $||Af||_q \leq c||f||_p$.

D: Such norm is *hypercontractive* when $p < q$.

D(*SDP hierarchy*): A relaxation of `SDP` into levels (rounds) where r rounds must be managable in time $n^{O(r)}$.

D: A functional $\tilde{E}$ that maps a polynomial P over $\mathbb{R}^n$ of degree at most r into a real number $\tilde{E}_x P(x)$ is a *level-r pseudo-expectation (functional)* if it satisfies:

- Linearity for polynomials of degree at most r,
- $\tilde{E}P^2 \geq 0$ for polynomials of degree at most $r/2$,
- $\tilde{E}1 = 1$.

D: Let $P_0, \ldots, P_m$ be polynomials over $\mathbb{R}^n$ of degree at most d, and lest $r \geq 2d$. The value of r-round SoS SDP for the program *max P_0 subject to $P_i^2 = 0$ for $i \in [m]$ is equal to the maximum of $\tilde{E}_{P_0}$ where $\tilde{E}$ ranges over all level r pseudo-expectation functionals satisfying $\tilde{E}P_i^2 = 0 \forall i \in [m]$.*

D: An algorithm provides a (c, C)-*aproximation* for the $2 \to q$ norm if for an operator A on input, the algorithm then can distunguish between the case that $||A||_{2\to q} \leq c\sigma$ and the case that $||A||_{2\to q} \geq C\sigma$, where σ is the minimum nonzero singular value of A.

T(*2.1*): For every $1 < c < C$, there is a $poly(n)exp(n^{2/q})$-time algorithm that computes a (c, C)-approximation for the $2 \to q$ norm of any linear operator whose range is $\mathbb{R}^n$.

T(*2.5, informal*): Assuming ETH, then for any ε, δ satisfying $\varepsilon + \delta < 1$ the $2 \to 4$ norm of an $m \times m$ matrix A cannot be approximated to within an m^εmultiplicative factor in time less than $m^{log^\delta(m)}$ time. This hardness result holds even with A being a projector.

T(*2.6, informal*): Eight rounds of the SoS relaxation certifies that it is possible to satisfy at most 1/100 fraction of the constraints in **Unique Games** instances of the "quotient noisy cube" and "short code" types.

The 2-to-q norm and small-set expansion

For simplicity, we consider only regular graphs.

D: *A measure* μ *of* $S \subseteq V(G)$ *will be* $|S|/|V|$. $G(S)$ will be the distribution obtained by picking a random $x \in S$ and then outputting a random neighbor y of x. Expansion $\Phi_G(S)$ can be then defined as $P_{y\in G(S)}[y \notin S]$.

We also identify G with its normalized adjacency matrix (adjacency matrix divided by d). The subspace $V_{\geq\lambda}(G)$ is defined as the span of eigenvectors of G with eigenvalue at least λ. The projector into such subspace will be denoted $P_{\geq\lambda(G)}$.

For a distribution D, we will use $cp(D)$ to denote the collision probability of D (that two indepedent samples from D are identical).

T(*2.4, equivalence*): For every regular graph G, $\lambda > 0$ and even q:

- (Norm bound implies expansion)
$$\forall\delta > 0, \varepsilon > 0, ||P_{\geq\lambda}(G)||_{2\to q} \leq \frac{\varepsilon}{\delta^{(q-2)/2q}}.$$
implies that $\Phi_G(\delta) \geq 1 - \lambda - \varepsilon^2$.
- (Expansion implies norm bound) There is a constant c such that
$$\forall\delta > 0, \Phi_G(\delta) > 1 - \lambda 2^{-cq}$$
implies that $||P_{\geq\lambda}(G)||_{2\to q} \leq \frac{2}{\sqrt{\delta}}$.

We will prove the second part of the theorem, as the previous one has already been proven before. We will require a few lemmas:

L(*Cheeger*): If $\Phi_G(\delta) \geq 1 - \eta$ then for all $f \in L_2(V)$ satisfying $||f||_1^2 \leq ||f||_2^2$ holds the following: $||Gf||_2^2 \leq c\sqrt{\eta}||f||_2^2$.

L: Let D be a distribution with $cp(D) \leq 1/N$ and g a function on a common ground set. Then $\exists T, |T| = N$ such that $E_{x\in T}[g(x)^2] \geq \frac{(E[g(D)])^2}{4}$.

The essence of the second part of the theorem is contained in the following lemma:

L(*Main lemma*): Set $e = e(\lambda, q) = 2^{cq}/\lambda$, with a constant $c \leq 100$. Then for every $\lambda > 0$ and $\delta \in [0, 1]$, if G is a regular graph that satisfies $cp(G(S)) \leq 1/(e|S|)$ for all S with $\mu(S) \leq \delta$, then $||f||_q \leq 2||f||_2/\sqrt{\delta}$ for all $f \in V_{\geq\lambda}(G)$.

We will use several claims throughout the proof of the Main lemma. They are stated here without specifying the various variables that will be context-bound.

Claim: Let $S \subseteq V$ and $\beta > 0$ be such that $|S| \leq \delta$ and $|f(x)| \geq \beta$ for all $x \in S$. Then there is a set T of size at least $e|S|$ such that $E_{x\in T}[g(x)^2] \geq \beta^2/4$.

Claim: $E_{x\in V}[g_j(x)^q] \geq e\alpha_{i_j}/(10c^2)^{q/2}$.

Claim(*The last claim*): $E_{x\in T}[g'_k(x)^2] \leq 100^{-i'}\beta_{i_j}^2/4$.