A bound for the number of vertices of a polytope with applications – *A. Barvinok*

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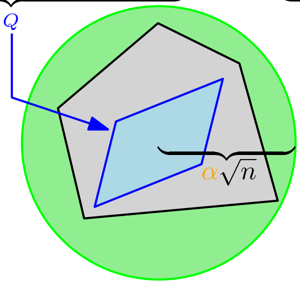
Theorem 1. $\forall \alpha$ there is $\gamma = \gamma(\alpha)$: $\forall m \le \alpha n$
$\forall u_1 \dots u_m \in \mathbb{R}^n$ where $\|u_i\| \le 1$, $\forall P \subseteq \mathbb{R}^n$ polytope
If $\underbrace{\{|\langle x, u_i\rangle| \le 1 \text{ for } i = 1, \dots, m\}}_{Q} \subseteq P \subseteq \{\|x\| \le \alpha\sqrt{n}\}$



then P has at least $2^{\gamma n}$ vertices.

Pf of Thm 1. Choose $\beta = \beta(\alpha)$ "big enough":
1)
2) ...

$$\Pr_y\left[\max_{x\in P}\langle y, x\rangle \ge \frac{n}{2\beta}\right] \ge \Pr_y\left[\begin{array}{c} y \in \beta Q \\ \|y\|^2 \ge \frac{n}{2}\end{array}\right] \ge \left(1 - e^{-\frac{\beta}{2}}\right)^{\alpha n} - e^{-\frac{n}{16}}$$

$$\|$$

$$\Pr_y\left[\max_{v\in W}\langle y, v\rangle \ge \tau\right] \le \sum_{v\in W}\Pr_y\left[\langle y, v\rangle \ge \tau\right] \le \frac{|W|}{2} e^{-\frac{\tau^2}{2\|v\|^2}} \le \frac{|W|}{2} e^{-\frac{n}{8\alpha^2\beta^2}}$$

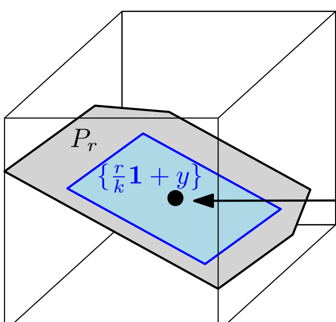
Lemma 1. Let $y \in \mathbb{R}^n$ be a random Gaussian vector. Then

$\forall n > 16$

1. $\Pr_y\left[\|y\|^2 < n/2\right] \le e^{-n/16}$
2. For arbitrary $u \in \mathbb{R}^n$ holds $\Pr_y[\langle u, y\rangle \ge \tau] \le e^{\tau^2/2\|u\|^2}$
3. For $u_1, \dots, u_m \in \mathbb{R}^n$, $\|u_i\| \le 1$ holds $\Pr_y[\langle u_i, y\rangle \le \beta] \ge \left(1 - e^{\beta^2/2}\right)^m \cdot (1/2)$

$P_r(G) \subseteq \mathbb{R}^{|E|}$ - the convex hull of all indicators χ_H where H are r-factors of a given $G = (V, E)$ is given by

$$\begin{array}{rcll} x_e & \in & [0,1] & \text{for all } e \in E \\ \sum_{e\in\delta(V)} x_e & = & r & \text{for all } v \in V \\ \sum_{e\in\delta(V)\setminus F} x_e - \sum_{e\in F} x_e & \ge & 1 - |F| & \text{for all } U \subseteq V, F \subseteq \delta(U) \text{ s. t. } r|U| + |F| \text{ is odd} \end{array}$$



Lemma 2. $\forall k, r > 0$ with $k \ge 2r + 1$ there is ϵ
Let G be a k-regular graph such that

$$|\delta(U)| > \frac{k}{r} \quad \text{for every } U \subseteq V \text{ such that } 2 \le |U| \le |V| - 2.$$

Then for all $y \in \mathbb{R}^E$ such that

$$\begin{array}{rcl} y_e & \in & [-\epsilon, \epsilon] \text{ for all } e \in E \\ \sum_{e\in\delta(v)} y_e & = & 0 \text{ for all } v \in V \end{array}$$

holds $\frac{r}{k}\mathbf{1} + y \in P_r(G)$

Theorem 2. $\forall k, r > 0$ with $k \ge 2r + 1$ there is $\gamma = \gamma(k, r)$ s. t.:
Let G be a k-regular graph such that

$$|\delta(U)| > \frac{k}{r} \quad \text{for every } U \subseteq V \text{ such that } 2 \le |U| \le |V| - 2.$$

Then the number of r-factors of G is at least $2^{\gamma|V|}$.