Rainbow Turán Problem for Even Cycles

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Rainbow Turán problem is a variant of an extremal graph theory problem which lies at the intersection of its two key areas. One hand we have classical Turán problem, which is still not solved in full generality for the case when the forbidden graph is bipartite. On the other hand, we have the Canonical Ramsey Theorem of Erdös and Rado which implies that every properly edge-colored K_n contains a rainbow copy of K_t, provided that n is large enough with respect to t.

1 Notation

In the whole presentation, the function log denotes the logarithm function of base 2, and ln denotes the natural logarithm. Next, all colorings that we consider are proper edge colorings. Finally, a rainbow copy of a fixed graph H in an edge-colored graph G is a copy of H in G such that no two edges of the copy have the same color.

2 Main Theorem

We use a nibbling-like argument to analyze a special random procedure in order to show the following main theorem.

Theorem 1. *For every fixed $\varepsilon > 0$ there is a constant $C := C(\varepsilon)$ such that every properly-colored graph on n vertices with at least $Cn^{1+\varepsilon}$ edges contains a rainbow copy of an even cycle of length at most $2k$, where $k = \left\lceil \frac{\ln 4 - \ln \varepsilon}{\ln(1+\varepsilon)} \right\rceil$.*

3 Probabilistic Tools

The main tools from the probability theory that we will use are appropriate variants of Chernoff/Hoeffding's inequalities and a union bound. It is a standard application of these tools to prove the following two lemmas.

Lemma 2. *Let G be an edge-colored graph on n vertices and minimum degree δ, let $\mathcal{C}$ be the set of colors in G, and let k be a positive integer. If*

$$nk \cdot \exp(-\delta/8k) < 1 \ ,$$

then there is a partition of $\mathcal{C}$ into k parts such that for every vertex v and every color class C, v is incident with at least $\delta/2k$ edges colored from C.

Lemma 3. *Let $\beta, \gamma \in (0,1)$ be fixed parameters. Suppose we have a set X and a collection of subsets X_j, where $j \in [m]$, such that $|X_j| \leq \beta|X|$ for each j. If*

$$3m \cdot \exp(-\beta \cdot \gamma|X|/8) < 1 \ ,$$

then there exists a subset $Y \subseteq X$ with $\gamma|X|/2 \leq |Y| \leq 2\gamma|X|$ such that for every $j \in [m]$, we have $|X_j \cap Y| \leq 4\beta|Y|$.