Partition Identities I : Sandwich Theorems and Logical 0-1 Laws

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Basic definitions

Partition identity is given by formula

$$\mathbb{A}(x) := \sum_{n=0}^{\infty} a(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}, \tag{1}$$

or, equivalently, by sequences $a(n)$ or $p(n)$, $n \in \mathbb{N}_0$, with setting $p(0) = 1$. We say that $a(n)$ is *partition (counting) function*, $p(n)$ is *component function* and $\mathbb{A}(x) := \sum_{n=0}^{\infty} a(n)x^n$ is *partition generating function*, shortly PGF. Set $\operatorname{rank}(p) := \sum p(n)$ the *rank* of PGF.

The property $f(n-1)/f(n) \to 1$ as $n \to \infty$, where $f(n)$ is eventually positive, is called RT_1. Similarly we define property RT_ρ for $\rho \in \mathbb{R}$.

A PGF $\mathbb{A}(x)$ is *reduced* if $\gcd\{n : p(n) > 0\} = 1$. If it is not, define $d = \gcd\{n : p(n) > 0\} > 1$ and PGF $\mathbb{A}^\star(x)$ with component function $p^\star(n) := p(nd)$ and counting function $a^\star(n) := a(nd)$. Then $\mathbb{A}^\star(x)$ is reduced and called *reduced form* of $\mathbb{A}(x)$.

Results

Theorem 1 (Bell) Let (1) be reduced partition identity. If $p(n)$ is polynomially bounded, that is, $p(n) = O(n^\gamma)$ for some $\gamma \in \mathbb{R}$, then $a(n)$ satisfies RT_1.

Theorem 2 (Bell and Burris) Suppose component function $p(n)$ satisfies RT_1, that is $p(n-1)/p(n) \to 1$ as $n \to \infty$. Then also partition function $a(n)$ satisfies RT_1.

Theorem 3 (Stewart's Sum Theorem) If for $j = 1, 2$ we have partition identity

$$\sum_{n=0}^{\infty} a_j(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-p_j(n)}$$

and each $a_j^\star(n)$ satisfies RT_1. Then also $a^\star(n)$ satisfies RT_1

$$\sum_{n=0}^{\infty} a(n)x^n = \prod_{n=1}^{\infty}(1-x^n)^{-p(n)}$$

with $p(n) = p_1(n) + p_2(n)$.

Theorem 4 Finite rank implies polynomial growth for $a(n)$. More precisely, if (1) is reduced and $r := \operatorname{rank}(p) < \infty$ then $a(n) \sim C \cdot n^{r-1}$ for some positive constant C.

Theorem 5 Infinite rank implies superpolynomial growth for $a(n)$. That is, if (1) is reduced and $r := \operatorname{rank}(p) = \infty$ then for all k we have $a(n)/n^k \to \infty$ as $n \to \infty$.

Theorem 6 (Schur) With $0 \le \rho < \infty$ suppose that $f(n)$ satisfies RT_ρ, $\mathbb{G}(x)$ has radius of convergence grater than ρ, and $\mathbb{G}(\rho) > 0$. Let $\mathbb{H}(x) = \mathbb{F}(x) \cdot \mathbb{G}(x)$. Then $h(n) \sim \mathbb{G}(\rho) f(n)$.

Sandwich theorem

For L nonnegative integer set $a^L(n) := a(n) + a(n-1) + \cdots + a(n-L)$. If PGF $\mathbb{A}$ has eventually positive coefficients, denote by $L_{\mathbb{A}}$ the least nonnegative integer L such that $a(n) > 0$ for all $n \ge L$.

Lemma 7 Let $\mathbb{A}(x)$ be a PGF with $a(n)$ eventually positive. Then for every $L \geq L_{\mathbb{A}}$ we have that $a^L(n)$ is nonincreasing for all n and positive for $n \geq L_{\mathbb{A}}$. Moreover $a^{mL}(n) \leq ma^L(n)$ for $m = 1, 2, \ldots$ and $n \geq 0$.

Lemma 8 Let $\mathbb{A}(x)$ be a PGF with $a(n)$ eventually positive. Suppose $L \geq L_{\mathbb{A}}$ is an integer such that $|a(n) - a(n-1)| = o(a^L(n))$. Then $a(n)$ satisfies RT_1.

Lemma 9 Suppose $\mathbb{A}_1(x)$ and $\mathbb{A}_2(x)$ are two PGFs and $L \geq L_{\mathbb{A}}$ a positive integer such that, with $\mathbb{A}(x) = \mathbb{A}_1(x) \cdot \mathbb{A}_2(x)$, $a_1(n)$ satisfies RT_1, and $a_2^L = o(a^L(n))$. Then $a(n)$ satisfies RT_1.

Theorem 10 (Sandwich Theorem) Suppose $\dot{\mathbb{A}}(x)$ is reduced PGF with $\dot{a}(n)$ satisfying RT_1. Then any

$$\mathbb{A}(x) := \sum_{n=0}^{\infty} a(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-p(n)}$$

satisfying $\dot{p}(n) \leq p(n) = O(\dot{a}(n))$ will be such that $a(n)$ satisfies RT_1.

Sandwich theorem can be used to prove Bell's Theorem 1. Following theorem is about necessity of conditions in Theorem 1.

Theorem 11 Let $f(n) \leq 1$ be a positive (nondecreasing) unbounded function. Then there is a PGF $\dot{\mathbb{A}}(x)$ satisfying RT_1 and component function $p(n)$ satisfying $\dot{p}(n) \leq p(n) = O(f(n)a(n))$, such that $a(n)$ fails to satisfy RT_1.

The notation $f(n) \leq g(n)$ menas that $f(n)$ is eventually less or equal to $g(n)$.

Theorem 12 (The Eventual Sandwich Theorem) Suppose $\dot{p}(n)$ satisfies RT_1 and $\dot{p}(n) \leq p(n) = O(\dot{a}(n))$. If

$$\sum_{n=1}^{\infty} (p(n) - \dot{p}(n)) \geq 0,$$

then $a(n)$ satisfies RT_1.

Logical 0–1 Laws

Monadic second order (MSO) logic for relational structures is just the usual first order logic augmented with variables and qualifiers for unary predicates.

Let $\mathcal{A}$ be a class of relational structures and $\mathcal{P}$ denote a subclass of connected structures. Class $\mathcal{A}$ is *adequate* if it is closed under disjoint union and extracting components. Therefore, if $\mathcal{A}$ is adequate then generating function $\mathbb{A}(x)$ is PGF (satisfies partition identity).

A class $\mathcal{A}$ of finite relational structures has a *MSO* 0–1 *law* if for every monadic second order sentence φ the probability that φ holds in randomly chosen member of $\mathcal{A}$ is either 0 or 1.

Let $a_{\mathcal{A}}(n)$ be the number of elements of $\mathcal{A}$ that have exactly n elements in their universe. Analogicaly, $p_{\mathcal{A}}(n)$ be the counting function for $\mathcal{P}$.

Theorem 13 (Compton) If $\mathcal{A}$ in an adequate class and $a_{\mathcal{A}}(n)$ satisfies RT_1 then $\mathcal{A}$ has a monadic second-order 0–1 law.

Theorem 10 gives us vast array of partition identities satisfying RT_1, and thus one has a correspondingly vast array of classes of relational structures with monadic second-order law.