Finitely forcible graphons

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Graphons and forcing families

Definition 1. Let $\mathcal{W}$ denote the set of bounded symmetric measurable functions of the form $W : [0,1]^2 \to \mathbb{R}$, and let $\mathcal{W}_0 \subset \mathcal{W}$ consist of those with range in $[0,1]$. The elements of $\mathcal{W}$ are called *graphons*.

Definition 2. The *density* $t(F,G)$ of a simple graph F in a simple graph G is the probability that a random map $V(F) \to V(G)$ is a graph homomorphism.
The *subgraph density* of a simple graph F in a graphon W is

$$t(G,W) = \int_{[0,1]^V} \prod_{ij \in E} W(x_i,x_j) \prod_{i \in V} dx_i.$$

Definition 3. Two graphons are *weakly isomorphic* if they have the same simple subgraph densities. We denote by $[W]$ the set of graphons weakly isomorphic to W.

Theorem 1. *Two graphons U and W are weakly isomorphic if and only if there are measure preserving maps $\varphi, \psi : [0,1] \to [0,1]$ such that $U^\varphi = W^\psi$, where $U^\varphi(x,y) := U(\varphi(x),\varphi(y))$.*

Definition 4. Let $F_1,\dots,F_k$ be simple graphs and $a_1,\dots,a_k$ be real numbers in $[0,1]$. We say that the set $\{(F_i,a_i)|i=1,\dots,k\}$ is a *forcing family* if there is a sequence of simple graphs $\{G_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} t(F_i,G_n) = a_i$ for every $i = 1,\dots,k$, and for every such graph sequence $\lim_{n\to\infty} t(F,G_n)$ exists for every simple graph F
Let $\mathcal{A} \subseteq \mathcal{W}$. Let $F_1,\dots,F_k$ be simple graphs and $a_1,\dots,a_k$ be real numbers in $[0,1]$. We say that the set $\{(F_i,a_i)|i=1,\dots,k\}$ is a *forcing family* in $\mathcal{A}$ if there is a unique (up to weak isomorphism) graphon $W \in \mathcal{A}$, such that $t(F_i,W) = a_i$ for every $i = 1,\dots,k$. In this case we say that W is *finitely forcible* in $\mathcal{A}$ and the family $\{F_i|i=1,\dots,k\}$ is a *forcing family for W in $\mathcal{A}$*.

Definition 5. A *quantum graph* is a formal linear combination with real coefficients of multigraphs. Multigraphs that occur with nonzero coefficients are called its *constituents*. A quantum graph is *simple* if every its constituent is simple. We denote a linear space of simple quantum graphs by $\mathcal{Q}$.

The adjoint of an operator

Definition 6. Let $\mathbf{F} : \mathcal{W} \to \mathcal{W}$ be an operator preserving weak isomorphism and let $\mathbf{F}^* : \mathcal{Q} \to \mathcal{Q}$ be a linear map. We say that the map $\mathbf{F}^*$ is an *adjoint of* $\mathbf{F}$ if

$$t(g,\mathbf{F}(W)) = t(\mathbf{F}^*(g),W)$$

for every $g \in \mathcal{Q}$ and $W \in \mathcal{W}$. We denote the set of functionals which have an adjoint by $\mathcal{D}$.

Lemma 2. *Let $W \in \mathcal{W}$ be finitely forcible in $\mathcal{W}$, let $\mathbf{F} \in \mathcal{D}$ and assume that $\mathbf{F}^{-1}([W])$ is finite (up to weak isomorphism). Then every element in $\mathbf{F}^{-1}([W])$ is finitely forcible in $\mathcal{W}$.*

Example 1. Let $\mathbf{F}(W) = \alpha W$ for some fixed $\alpha \in \mathbb{R}$. Then $\mathbf{F}^*$ for simple graphs is

$$\mathbf{F}^*(G) = \alpha^{|E(G)|} G.$$

Example 2. Let $\mathbf{F}(W) = W + \beta$ for some fixed $\beta \in \mathbb{R}$. Then $\mathbf{F}^*$ for simple graphs is

$$\mathbf{F}^*(G) = \sum_{Z \subseteq E(G)} \beta^{|E(G) \setminus Z|} (V(G), Z).$$

Corollary 3. *If $W \in \mathcal{W}$ is finitely forcible (in $\mathcal{W}$), then $\alpha W + \beta$ for $\alpha, \beta \in \mathbb{R}$ is finitely forcible.*

Necessary condition for finite forcing

Definition 7. A graphon W is a *stepfunction* if there is a partition $\{S_1, \dots, S_n\}$ of $[0,1]$ into measurable sets such that W is constant on each product set $S_i \times S_j$.

Definition 8. We say that the *rank* of a graphon W is r, if r is the least nonnegative integer such that there are measurable functions $w_i : [0,1] \to \mathbb{R}$ and reals λ_i, $i = 1, \dots, r$, such that

$$W(x,y) = \sum_{k=1}^{r} \lambda_k w_k(x) w_k(y)$$

almost everywhere. If no such integer exists, then we say that W has infinite rank.

Theorem 4. *If W has finite rank, then for every finite list $F_1, \dots F_m$ of simple graphs there is a stepfunction U such that $t(F_i, U) = t(F_i, W)$ for every $i = 1, \dots, m$.*

Corollary 5. *Every finitely forcible graphon is either a stepfunction or it has infinite rank.*

Corollary 6. *Assume that $W \in \mathcal{W}$ can be expressed as a non-constant polynomial in x and y. Then W is not finitely forcible.*

Finitely forcible graphons

Definition 9. Suppose that edges of a graph F are partitioned into two sets E_+ and E_-. We call the triple $\widehat{F} = (V, E_+, E_-)$ a *signed graph* and we define

$$t(\widehat{F}, W) = \int_{[0,1]^V} \prod_{ij \in E_+} W(x_i, x_j) \prod_{ij \in E_-} (1 - W(x_i, x_j)) \prod_{i \in V} dx_i.$$

Definition 10. A graph F with k specified vertices labeled $1, \dots, k$ and any number of unlabeled vertices is called a *k-labeled graph*. Let V_0 be a set of unlabeled vertices of F. For $W \in \mathcal{W}$ we define a function $t_k(F, W) : [0,1]^k \to \mathbb{R}$ by

$$t_k(F, W)(x_1, \dots, x_k) = \int_{[0,1]^{V_0}} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V_0} dx_i.$$

Let $\mathcal{M}_0$ denote the set of functions $[0,1]^2 \to [0,1]$ that are monotone decreasing in both variables, and let $\mathcal{M}$ be the set of graphons which are weakly isomorphic to some function of $\mathcal{M}_0$. Let $\widehat{C}_4$ denote a signed 4-labeled 4-cycle, with two opposite edges signed "+"and the other two "-".

Lemma 7. *Let $W \in \mathcal{W}$, then $W \in \mathcal{M}$ if and only if $t_4(\widehat{C}_4, W) = 0$ almost everywhere.*

Theorem 8. *Let p be a real symmetric polynomial in two variables, which is monotone decreasing on $[0,1]$. Then the function $W(x,y) = \mathbf{1}_{p(x,y) \geq 0}$ is finitely forcible in $\mathcal{W}$.*