ON THE CHROMATIC NUMBERS OF SPHERES IN $\mathbb{R}^n$

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Definitions. the *chromatic number* of a set $X \subseteq \mathbb{R}^n$:

$$\chi(X) = \min\{c; X = X_1 \cup X_2 \cup \dots X_c, \forall i\ \forall x, y \in X_i \quad |x - y| \neq 1\}$$
$$S_r^{n-1} = \{x \in \mathbb{R}^n; |x| = r\}$$

Known results:

- $4 \leq \chi(\mathbb{R}^2) \leq 7$
- $\chi(\mathbb{R}^n) \leq (3 + o(1))^n$ [Larman and Rogers, 1972]
- $\chi(\mathbb{R}^n) \geq (1.207\dots + o(1))^n$ [Frankl and Wilson, 1981]
- $\chi(\mathbb{R}^n) \geq (1.239\dots + o(1))^n$ [Raigorodskii, 2000]
- $\chi(S_r^{n-1}) \leq cn^{5/2}(2r)^n$, if $r > 1/2$ [Rogers, 1963]

Conjecture. [Erdős, 1981] $\chi(S_r^{n-1}) \to \infty$ for any fixed $r > 1/2$.

Claim. [Lovász, 1983] $\chi(S_r^{n-1}) \geq n$ for $r > 1/2$ and $\chi(S_r^{n-1}) \leq n+1$ for $r < \sqrt{\frac{n}{2n+2}}$.

Conjecture. [Lovász, 1983] $\chi(S_r^{n-1})$ grows exponentially for $r > \sqrt{\frac{n}{2n+2}}$.

Theorem 1. For any $r \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, there exists a function $\delta(n) = \delta(n, r) = o(1)$, $n \to \infty$, such that for every $n \in \mathbb{N}$, we have

$$\chi(S_r^{n-1}) \geq \left(2q^q(1-q)^{1-q} + \delta(n)\right)^n,$$

where $q = \frac{1}{8r^2}$.

Proof. (*sketch*) Consider the following graph $G(W, F)$:

$$W = \{x = (x_1, \dots, x_m); x_i \in \{-1, 1\}, x_1 + \dots + x_m = 0\},$$

$$F = \{\{x, y\}; x, y \in V, |x - y| = \sqrt{2m - 2a}\},$$

where $m = 4\lfloor \frac{n-1}{4} \rfloor$, $r = \frac{\sqrt{m}}{\sqrt{2m-2a'}}$,
p is the smallest prime number satisfying $p > \frac{m-a'}{4}$ and $a = m - 4p < a'$.
Estimate $\alpha(G)$ by investigating the set of polynomials

$$P'_x(y) = \prod_{i \in \{0,1,\dots,p-1\} \setminus \{m \bmod p\}} (i - (x, y)), \quad x \in W.$$

$\square$

Theorem 2. Let $\mathbb{P}$ be the set of prime numbers. Let $f(x)$ be such a function that for any $x \in \mathbb{R}$, $x \geq 0$,

$$f(x) = \min\{p \in \mathbb{P} : p > x\} - x.$$

Let

$$m(x) = \max\{m < x; m \equiv 0 \pmod 4\}.$$

Consider a sequence $\{r_n\}_{n=1}^{\infty}$, where $r_n > \frac{1}{2}$ for each $n \in N$. Set

$$p(n) = \frac{m(n)}{8r_n^2} + f\left(\frac{m(n)}{8r_n^2}\right).$$

If

$$\frac{m(n)}{4} \leq p(n) \leq \frac{m(n)}{2}, \quad n \in \mathbb{N},$$

then

$$\chi(S_r^{n-1}) \geq \frac{\binom{m(n)}{m(n)/2}}{\binom{m(n)}{p(n)}}.$$

Theorem 3. Consider a sequence $\{r_n\}_{n=1}^{\infty}$, where $r_n > \frac{1}{2}$ for each $n \in N$. Let $\kappa < 2$, and let $p(n)$ be the same as in Theorem 2. If

$$\frac{m(n)}{4} \leq p(n) \leq \frac{m(n)}{2} - \sqrt{\frac{m(n)\ln m(n)}{\kappa}}, \quad n \in \mathbb{N},$$

then

$$\chi(S_r^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

Theorem. [Baker, Harman and Pintz, 2001] The "prime gap" function satisfies

$$f(x) = O(x^{0.525}).$$

Theorem 4. Assume that $c_0 > 0$ is such that $f(x) \leq c_0 x^{0.525}$ for every x. Then, there exists a constant $c_0' > 0$ such that for any sequence of radii r_n satisfying the inequality

$$r_n \geq \frac{1}{2} + \frac{c_0'}{n^{0.475}},$$

we have the bound

$$\chi(S_r^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

Theorem 5. Assume that $c_1 > 0$ is such that $f(x) \leq c_1 \ln^2 x$ for every x. Then, there exists a constant $c_1' > 0$ such that for any sequence of radii r_n satisfying the inequality

$$r_n \geq \frac{1}{2} + c_1' \sqrt{\frac{\ln n}{n}},$$

we have the bound

$$\chi(S_r^{n-1}) > n + 1, \quad \forall n \geq n_0.$$

Theorem 6. There exists a constant $c_2 > 0$ such that for any sequence of radii r_n satisfying the inequality

$$r_n \leq \frac{1}{2} + \frac{c_2}{n}$$

we have the bound

$$\chi(S_r^{n-1}) \leq n + 1, \quad \forall n \geq n_0.$$