Approximate Center Points with Proofs

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Definition (Centerpoint). *A centerpoint of a set $S \subset \mathbb{R}^d, |S| = n$ is a point c such that every closed half-space containing c also contains at least $\frac{n}{d+1}$ points of S.*

Definition (β-center). *Given a set S. Point $c \in S$ is called β-center if every closed half-space containing c also contains at least β fraction of the points of S.*

Theorem (Radon, 1921). *Given $n > d+1$ points $S \subset \mathbb{R}^d$, there exist a partition $(U, \overline{U})$ of S such that* $\operatorname{conv}(U) \cap \operatorname{conv}(\overline{U}) \neq \emptyset$.

$\Rightarrow$ Radon partition + Radon point.

Theorem (The centerpoint theorem). *Given a set of n points $S \subset \mathbb{R}^d$, there exist a centerpoint $c \in \mathbb{R}^d$ such that every closed half-space containing c also contains at least $\lceil \frac{n}{d+1} \rceil$ points of S.*

Theorem (Helly, 1913). *Given a collection of compact, convex sets $X_1, X_2, \ldots, X_n \subset \mathbb{R}^d$. If every $d+1$ of these sets have a common intersection, then the whole collection has a common intersection.*

Theorem (Tverberg, 1966). *Given $(d+1)(r-1)+1$ points $S \subset \mathbb{R}^d$, there exists a partition of S into $S_1, S_2, \ldots, S_r$, such that* $\bigcap_{i=1}^{r} \operatorname{conv}(S_i) \neq \emptyset$.

$\Rightarrow$ Tverberg partition + Tverberg point + Depth of Tverberg point.

Theorem (Carathéodory, 1911). *Given a point $c \in \mathbb{R}^d$ lying in the convex hull of a set P. If P has more than $d+1$ points, then there is a subset $P^* \subset P$ of size $d+1$ and c lies in the convex hull of P^*.*

Lemma 1. *Given a set P of $d+2$ Tverberg points of depth r with disjoint partitions, the Radon point of P has depth $2r$.*

Lemma 2. *If there is a proof that a point p has depth r, there exists such a proof that contains at most $r(d+1)$ points of S.*

Theorem (Iterated-Tverberg analysis). *Given n points in $\mathbb{R}^d$, the Iterated-Tverberg algorithm always returns a $\frac{n}{2(d+1)^2}$-center.*