3-SAT Faster and Simpler - $O(1.308^n)$ - *T. Hertli*

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$\underline{V}$ denotes set of **variables**
Literal $\underline{\ell}$ over $x \in V$ is either x or $\bar{x}$
Clause over V is a set of literals over V.
Formula in (k-)CNF form $\mathbf{F}$ is $(F, V = V_{\mathbf{F}})$ where F is set of clauses over V (of cardinality $\leq k$).

Truth assignment (přiřazení) $\underline{\alpha}$ is a function $V \to \{0,1\}$. It is **satisfying** iff ….
$\underline{\mathrm{sat}_F}$ is the set of all satisfying assignments.
$\underline{\mathbf{F}^{[x \mapsto \alpha]}}$ denotes $\mathbf{F}$ with x substituted by $\alpha \in \{0,1\}$.
$\underline{\mathbf{F}^{[x]}}$, resp. $\underline{\mathbf{F}^{[\bar{x}]}}$ denotes $\mathbf{F}^{[x \mapsto \alpha]}$ when $\alpha = 1$, resp. 0.

(For the rest, fix an integer $s > 0$.) Literal l is (s-)**implied** iff $\exists G \subseteq F$ of cardinality $\leq s$ such that $l \in \alpha$ for all $\alpha \in \mathrm{sat}_{(G, V_{\mathbf{F}})}$

procedure PPSZ($\mathbf{F}$)
  Choose β u.r. assignment on $V_{\mathbf{F}}$
  Choose π u.r. permutation of $V_{\mathbf{F}}$
  Partial assignment $\alpha := \emptyset$
  while $V_{\mathbf{F}} \neq \emptyset$ **do**
    while $\exists s$-implied $l = y \mapsto a$ **do**
      $\mathbf{F} := \mathbf{F}^{[l]}$
      $\alpha := \alpha \cup \{l\}$
    x :=the first variable of $V_{\mathbf{F}}$ in π
    $\mathbf{F} := \mathbf{F}^{[x \mapsto \beta(x)]}$
    $\alpha := \alpha \cup \{x \mapsto \beta(x)\}$
  return α

PPSZ$_{\beta,\pi}(\mathbf{F})$

such x is called **guessed (hádaná)**
such y is called **forced (vynucená)**
(in $PPSZ_{\beta,\pi}(\mathbf{F})$)

$\underline{p_{\text{success}}(\mathbf{F})} := \Pr_{\beta,\pi}[\mathrm{PPSZ}_{\beta,\pi}(\mathbf{F}) \in \mathrm{sat}_{\mathbf{F}}]$

for $\alpha \in \mathrm{sat}_{\mathbf{F}}$ $\underline{p_{\text{guessed}}(\mathbf{F}, x, \alpha)} :=$

$$\Pr_{\pi}[x \text{ is guessed in } \mathrm{PPSZ}_{\alpha,\pi}]$$

Lemma 10. For $\ell \in \alpha$, $\alpha \in \mathrm{sat}$
$p_{\text{guessed}}(\mathbf{F}^{[l]}, x, \alpha) \leq p_{\text{guessed}}(\mathbf{F}, x, \alpha)$

set of **frozen (zmrzlých)** variables $\underline{V_{\mathbf{F}}^{-}} := \{x \mapsto a \text{ is in every } \alpha \in \mathrm{sat}_{\mathbf{F}} \text{ for some fixed } a\}$, $|V_{\mathbf{F}}^{-}| = \underline{n_{\mathbf{F}}^{-}}$
set of **non-frozen (nezmrzlých)** variables $\underline{V_{\mathbf{F}}^{-}} := V_{\mathbf{F}} \setminus V_{\mathbf{F}}^{-}$, $|V_{\mathbf{F}}^{+}| = \underline{n_{\mathbf{F}}^{+}}$
set of **satisfying literals** $\underline{\mathrm{SL}_F} := \{\ell \text{ literal} \mid \mathbf{F}^{[\ell]} \text{ is satisfiable}\}$

procedure AssignSL($\mathbf{F}$)
(random process)
  $\alpha := \emptyset$
  while $V_{\mathbf{F}} \neq \emptyset$ **do**
    Choose $\ell \in \mathrm{SL}_{\mathbf{F}}$
    $\alpha := \alpha \cup \{\ell\}$
    $\mathbf{F} := \mathbf{F}^{[\ell]}$
  return α

$\underline{p(\mathbf{F}, \alpha)} := \Pr[\mathrm{AssignSL}(F) = \alpha]$

$S := S_k + \epsilon_k(s)$

$$\underline{c_{\mathbf{F},x}} := \begin{cases} S, & x \in V_{\mathbf{F}}^{+}; \\ \sum_{\alpha \in \mathrm{sat}_{\mathbf{F}}} p(\mathbf{F}, \alpha) p_{\text{guessed}}(\mathbf{F}, x, \alpha), & x \in V_{\mathbf{F}}^{-} \end{cases}$$

$$\underline{c_{\mathbf{F}}} := \sum_{x \in V_{\mathbf{F}}} c_{\mathbf{F},x}$$

Lemma 16. $\ell' = x' \mapsto \alpha(x') \in \alpha, \alpha \in \mathrm{sat}_{\mathbf{F}}$:

$$\begin{array}{rcl} p(\mathbf{F}^{[\ell']}, \alpha) & \geq & p(\mathbf{F}, \alpha) \\ \text{if } x' \text{ is frozen} & = & \\ \text{and } c_{\mathbf{F}}^{[\ell']} & \leq & c_{\mathbf{F}} \end{array}$$

Theorem 15. $p_{\text{success}}(\mathbf{F}) \geq 2^{-c_{\mathbf{F}}}$.

Theorem 17. $\forall \mathbf{F}$ s-implication free, $\ell \in \mathrm{SL}_{\mathbf{F}}$:

$$E_{\ell}[c_{\mathbf{F}^{[\ell]}}] \leq c_{\mathbf{F}} - n_{\mathbf{F}}^{+} \frac{2S}{|\mathrm{SL}_{\mathbf{F}}|} - n_{\mathbf{F}}^{-} \frac{1}{|\mathrm{SL}_{\mathbf{F}}|}$$

procedure PPSZ'($\mathbf{F}$)
  $\alpha := \emptyset$
  Use implications
  Choose $x \in V_{\mathbf{F}}$
  Choose $a \in \{0,1\}$
return $\alpha \cup \{x \mapsto a\} \cup \mathrm{PPSZ'}(\mathbf{F}^{[x \mapsto a]})$