Planarity Testing Revisited

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Definition 1. *The bridges of a cycle C consist of:*

- *For every connected component X of $G \setminus C$, the induced graph $G[X \cup A_X]$ where $A_X \subseteq C$ are the vertices of C adjacent to some vertex of X (the so called points of attachment).*
- *The chords of C (here the endpoints of C are its points of attachment)*

Definition 2. *Two bridges B_1, B_2 of a cycle C conflict iff either of the following conditions hold:*

- *a_i, a'_i are two points of attachment of B_i w.r.t. C for $i \in 1,2$ such that they occur in the order a_1, a_2, a'_1, a'_2 along the cycle C.*
- *B_1, B_2 have three common points of attachment w.r.t. cycle C.*

Definition 3. *Given a spanning tree T of a biconnected graph G, and an edge $e \in E(G) \setminus E(T)$, the graph $T \cup e$ contains a unique cycle $C(e)$ called the fundamental cycle of e. We say that a face of an embedded planar graph is fundamental if it is a fundamental cycle of some non-tree edge (with respect to some fixed spanning tree).*

Fact 1 *The list of edges in each fundamental cycle of G w.r.t. a spanning tree T can be obtained by a logspace transducer.*

Fact 2 *The faces of a 3-connected planar graph G are exactly the induced non-separating cycles of G. Further, a 3-connected graph is planar iff every edge lies on exactly two induced, non-separating cycles.*

Proposition 1. *Given a the cyclic order of vertices in every face of a biconnected embedded graph, it is possible to construct in logspace, a combinatorial embedding of the graph.*

Lemma 1. *The triconnected components of a graph can be obtained in logpace.*

Lemma 2. *Given a combinatorial planar embedding of the triconnected components of a graph, it is possible to obtain the biconnected planar embedding of the graph in logspace*

Lemma 3. *Given a 3-connected planar graph G and an arbitrary cycle C in the graph the conflict graph $H_C(G)$ is bipartite and connected.*

Proposition 2. *Any biconnected embedded planar graph has a fundamental face (i.e. a fundamental cycle which is also a face) w.r.t. each of its spanning trees.*

Corollary 1. *Every 3-connected planar graph has at least one fundamental face and this can be found by a Logspace transducer.*

Definition 4. *For distinct non-tree edges e_1, e_2 , define $e_1 \prec e_2$ iff in a 2-coloring of the conflict graph $H_{C(e_2)}(G)$, the colors of the vertices corresponding to bridges containing e_0, e_1 get different colors.*

For each $e \in E(G) \setminus \{e_0\}$ we define:

$$P(e) = \{e' \in E(G) \setminus (E(T) \cup \{e\}) | e' \prec e \& \neg \exists e'' : e' \prec e'' \prec e\}$$

Lemma 4. *For each non-tree edge $e = e_0$, the face $f(e)$ consists exactly of the edges in the set $F(e)$.*

Theorem 3. *Given a graph G, constructing a planar embedding for G if it is planar and otherwise rejecting it, can be done in logspace.*