The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$

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Definitions

- $\mathbb{H}$ = hexagonal lattice
- H = set of *mid-edges* of $\mathbb{H}$
- $\gamma : a \to b$ = a walk γ from the mid-edge a to the mid-edge b
- $\gamma : a \to E$ = a walk γ from the mid-edge a to a mid-edge from E
- $\ell(\gamma)$ = the length of the walk γ = the number of vertices on γ
- c_n = number of self-avoiding walks on $\mathbb{H}$ of length n starting from the origin

Observation.

- $\sqrt{2}^n \le c_n \le 4 \cdot 2^{n-1}$
- $c_{n+m} \le c_n c_m$

Lemma (Fekete). For a subadditive nonnegative sequence a_n the limit $\lim_{n\to\infty} a_n/n$ exists.

Corollary. There exists $\mu \in (1, \infty)$ such that $\mu = \lim_{n\to\infty} c_n^{1/n}$.

Theorem. $\mu = \sqrt{2+\sqrt{2}} = 2\cos(\pi/8)$.

Proof of the Theorem

$$Z(x) := \sum_{n=0}^{\infty} c_n x^{-n} = \sum_{\gamma\,:\,a\to H} x^{-\ell(\gamma)} \qquad \in (0, +\infty]$$

idea: show that $Z(x) = +\infty$ for $x < \sqrt{2+\sqrt{2}}$ and $Z(x) < +\infty$ for $x > \sqrt{2+\sqrt{2}}$.

- $x_c := \sqrt{2+\sqrt{2}}$, $\sigma := 5/8$, $j := \mathrm{e}^{\mathrm{i}2\pi/3}$
- a *domain* $\Omega \subseteq H$ = a (simply connected) union of all mid-edges adjacent to a subset $V(\Omega)$ of vertices of $\mathbb{H}$.
- $\partial\Omega$ = a set of mid-edges adjacent to only one vertex from $V(\Omega)$
- $\mathrm{W}_\gamma(a, b)$ = *winding* of γ between mid-edges a and b ($+\pi/3$ for each left turn, $-\pi/3$ for each right turn)
- (*"parafermionic observable"*) for $a \in \partial\Omega$ and $z \in \Omega$,

$$F(z) := F(a, z, x, \sigma) := \sum_{\gamma\subset\Omega\,:\,a\to z} \mathrm{e}^{-\mathrm{i}\sigma \mathrm{W}_\gamma(a,z)} x^{-\ell(\gamma)}.$$

Lemma 1. If $x = x_c$ and $\sigma = \frac{5}{8}$, then F satisfies the following relation for every vertex $v \in V(\Omega)$:

$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) = 0,$$

where p, q, r are the mid-edges of the three edges adjacent to v (in counter-clockwise order).

Counting in the strip and the trapezoid domain

vertical strip domain S_T: $V(S_T) = \{z \in V(\mathbb{H}) : 0 \leq \mathrm{Re}(z) \leq \frac{3T+1}{2}\}$

trapezoid domain $S_{T,L}$: $V(S_{T,L}) = \{z \in V(S_T) : |\sqrt{3}\mathrm{Im}(z) - \mathrm{Re}(z)| \leq 3L\}$

$$A^x_{T,L} := \sum_{\gamma \subset S_{T,L}:\ a \to \alpha \setminus \{a\}} x^{-\ell(\gamma)},\ B^x_{T,L} := \sum_{\gamma \subset S_{T,L}:\ a \to \beta} x^{-\ell(\gamma)},\ E^x_{T,L} := \sum_{\gamma \subset S_{T,L}:\ a \to \varepsilon \cup \bar{\varepsilon}} x^{-\ell(\gamma)}.$$

Lemma 2. When $x = x_c$, we have

$$\cos(3\pi/8) A^{x_c}_{T,L} + B^{x_c}_{T,L} + \cos(\pi/4) E^{x_c}_{T,L} = 1.$$

Observation. For $x \geq x_c$, the following limits exist and are finite:

$$\begin{aligned} A^x_T &= \lim_{L \to \infty} A^x_{T,L} = \sum_{\gamma \subset S_T:\ a \to \alpha \setminus \{a\}} x^{-\ell(\gamma)}, \\ B^x_T &= \lim_{L \to \infty} B^x_{T,L} = \sum_{\gamma \subset S_T:\ a \to \beta} x^{-\ell(\gamma)}, \\ E^{x_c}_T &= \lim_{L \to \infty} E^{x_c}_{T,L}. \end{aligned}$$

Corollary. $\cos(3\pi/8) A^{x_c}_T + B^{x_c}_T + \cos(\pi/4) E^{x_c}_T = 1.$

I) Proof that $Z(x_c) = +\infty$

II) Proof that $Z(x) < +\infty$ for $x > x_c$